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XXXVII. *Theory of Heat Conduction and Convection from Short-Hot Vertical Cylinders.* *By* W. S. KIMBALL, *Ph.D.*, *M.A.*, *and* L. D. CHILDS *.

1. *Introduction.*

AN analysis has recently been made † of the heat transfer by free conduction and convection from hot vertical plates of small height in air. The present paper extends this same theory to include short-hot vertical cylinders. It is assumed, as in the hot vertical plate study, that heat transfer takes place by pure conduction, through gas in stream-line motion. This assumption restricts the applicability of the present theory, as well as the previous vertical plate case, to surfaces not more than about two feet high. For experiment ‡ shows that above this limiting height turbulence sets in, and the state of affairs is quite different from that described here.

Below this limiting height, however, the analysis checks § all the available data to a highly acceptable degree of accuracy.

A short cylinder whose surface is maintained at a hot uniform temperature stands vertically in an atmosphere

* Communicated by the Authors.

† W. S. Kimball and W. J. King, Phil. Mag. xiii. p. 888 (May 1932).

‡ E. Griffiths and A. H. Davis, Rept. of Food Invest. Bd., H.M. Stationery Office (1922).

§ Kimball and King, *loc. cit.*

of air at ordinary room temperature. Surrounding this cylinder and right next to it is a thin film of almost stationary gas through which heat flows by conduction and convection.

Two empirical laws plus the classical equations of hydrodynamics and conduction, form the basis of this study :

Law I.—The locus of the maxima of the velocity curves is an isothermal surface at temperature $T_m = \frac{T_1+T_0}{2}$ halfway between that of the hot surface and the ambient air.

Apparently no measurements of temperature drop and convection velocity around vertical hot cylinders have been made, since no such data appears in the literature. Yet the assumption that this law applies to vertical cylinders as well as vertical plates has some experimental basis, *i. e.* (*a*) we herein derive the well known five-fourths power law for variation of heat transfer with temperature excess, and other formulas that agree with experiment, and (*b*) cylinders of large radius have surfaces approximating a plane, for which these laws are experimentally justified *.

Law II.—Half the heat is convected up away inside the film and half outside. This film is the isothermal surface of half temperature drop, which is the locus of the maxima of the velocity curves according to Law I. Like the previous experimental law, Law II., as applied to cylinders, is justified by indirect tests and the approximate equivalence of large cylindrical surfaces to plane surfaces.

2. *Temperature within the Film.*

Inside the film, we take as the empirical expression for temperature

$$T = T_1 - a \log \frac{r}{r_0}(1+be^{-\alpha y}), \quad . \quad . \quad . \quad . \quad (1)$$

where T_1 is the temperature of the hot cylinder, r_0 is the radius of the hot cylinder, and r is the distance from any point within the film to the axis of the cylinder, and a, b, α are experimental constants.

The radius of the film boundary is given by using Law I. and substituting $T = \frac{T_1+T_0}{2}$ in (1), and solving for r :

$$\log \frac{r_m}{r_0} = \frac{T_1-T_0}{2a(1+be^{-\alpha y})}; \quad r_m = r_0\left(\exp \frac{T_1-T_0}{2a(1+be^{-\alpha y})}\right), \quad (2)$$

* Kimball and King, *loc. cit.*

and the film thickness by

$$r_m - r_0 = r_0\left(\exp\frac{T_1 - T_0}{2a(1+be^{-ay})} - 1\right). \quad . \quad . \quad . \quad (3)$$

3. *The Langmuir Film and the Isothermal Film.*

It is to be noted that the above isothermal film, which by Law I. is the locus of the maxima of the velocity curves, is not the same as the familiar Langmuir Film. The gist of the Langmuir Film Theory is that heat transfer laws, etc., are exactly what they would be in case of pure conduction across a stationary film of gas, with outer boundary at the ambient temperature T_0 and having uniform thickness determined by the measurable physical constants that apply. It deals with resultant effects and does not purport to include the details of the mechanism of heat transfer across the film of "stationary" gas surrounding the hot cylinder. Thus the heat transferred from a cylinder L-units long is :

$$Q = -2\pi KLr\frac{dT}{dr}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where K is the conductivity coefficient, and it will be noted that it is the temperature alone that varies with r. Now integrate between the limits r_0 and r_L and the corresponding T_1 and T_0 for temperature. Then the r_L so determined is the Langmuir film radius by definition :

$$\log\frac{r_L}{r_0} = \frac{2\pi KL(T_1 - T_0)}{Q} = \frac{(T_1 - T_0)}{c}, \quad . \quad . \quad (5)$$

and the temperature within the film is

$$T = T_1 - c\log\frac{r}{r_0}; \qquad c = \frac{Q}{2\pi KL}. \quad . \quad . \quad . \quad (6)$$

Comparison of (1) and (6) shows that (1) also represents a temperature distribution corresponding to pure conduction, except for the factor $(1+be^{-ay})$, just as in the case of the hot *plane* surface the actual temperature :

$$T = T_1 - ax(1+be^{-ay})$$

gave exactly the form of distribution for pure conduction except for the factor $(1+be^{-ay})$.

Furthermore, we shall see that the constant c of (6) equals $2a$ of (1), so that the Langmuir film thickness and boundary determined by (5) is the same as the isothermal film boundary in the upper regions where the exponential

terms of (2) and (3) are negligible. See fig. 1 ; see also Section IX. below.

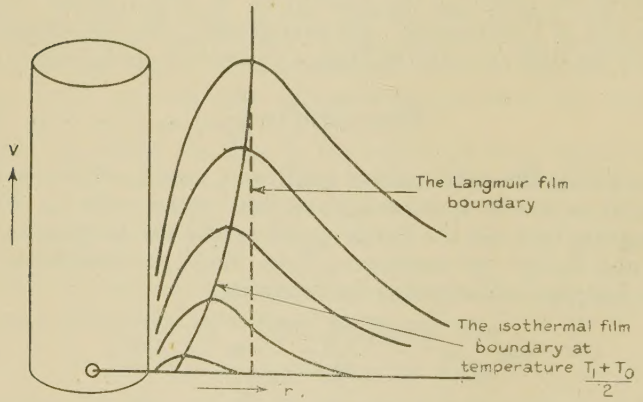
4. *The Classical Heat Equation.*

Inside the film the temperature expression (1) satisfies the differential equation :

$$-\mathrm{K}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathrm{T}}{\partial r}\right)+\left(\frac{\partial^2 \mathrm{T}}{\partial y^2}\right)=c\frac{\partial \mathrm{T}}{\partial y}, \quad . \quad . \quad (7)$$

which differs from the simplified classical hydrodynamic equation for combined conduction and convection, only in that $c=\alpha \mathrm{K}$, approximately a constant instead of the convection energy flux density per degree. This latter is :

Fig. 1.



Law I. illustrated by velocity curves with maxima on the isothermal film.

$c=\frac{3}{2}knv$, where k is Boltzmann's constant, v the convection velocity, and n the molecular density, and it varies from zero at the hot surface where $v=0$ to a maximum at the film boundary where v is maximum.

No experimental data on temperature fields around cylinders are available as such to check this temperature expression and justify this departure from classical theory. Yet we know that, for cylinders of large radius, it is correct since these approximate plane surfaces, and for them

$$\mathrm{T}=\mathrm{T}_1-a\log\frac{r}{r_0}(1+be^{-\alpha y})$$

reduces to

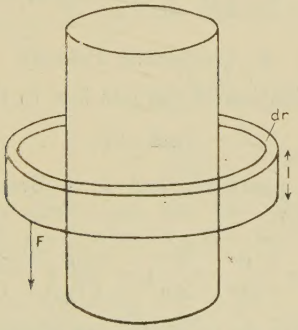
$$\mathrm{T}=\mathrm{T}_1-ax(1+be^{-\alpha y})$$

which is derived directly from experiment*. Thus the present temperature expression introduces the same departures from classical theory in the cylindrical case that were introduced into the plane case, and hence are surely correct and experimentally justified at least for large cylinders. It merely extends the previous hot plate theory to include hot cylinders, with no change except that required by the change from cartesian to cylindrical coordinates.

5. *The Force Equation.*

Within the film, the fundamental equations of hydrodynamics are simplified by neglecting inertia effects, second order velocity effects, horizontal velocities, and by using the gas law. The downward viscous force F on the annular

Fig. 2.



inside surface of dimensions $2\pi r \times 1$ will then be (see fig. 2):

$$F = -2\pi\eta r \frac{dv}{dr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where η is the viscosity coefficient of the gas, and v is its vertical convection velocity. The difference between the force outside this ring of thickness dr and the force inside is a resultant downward force which is the viscous drag†. This is given by

$$dF = +2\pi\eta \frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)dr. \quad . \quad . \quad . \quad . \quad (9)$$

If there is no acceleration, this force must balance the buoyancy. Let m be the mass of a molecule of air,

* Kimball and King, *loc. cit.*

† Leigh Page, 'Introduction to Theoretical Physics,' p. 230.

g the acceleration of gravity, n the molecular density of the hot gas, and n_0 that of the cooler gas. Then mgn is the weight in absolute units per unit volume of the hot gas, and mgn_0 the corresponding weight of the displaced unit volume of the cooler gas. From Archimedes's principle, the buoyant force per unit volume is

$$mgn_0 - mgn = mg(n_0 - n). \quad . \quad . \quad . \quad . \quad (10)$$

The condition that the buoyancy of the ring is balanced by the viscous drag on it is, therefore,

$$-2\pi\eta \frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)dr = 2\pi mg(n_0 - n)r\,dr,$$

from which we obtain the fundamental force equation :

$$-\eta \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = mg(n_0 - n). \quad . \quad . \quad . \quad (11)$$

6. *Convection Velocity.*

With the application of the gas law in the form

$$p = nk\mathrm{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where p is the pressure, and k is Boltzmann's constant, eq. (11) becomes :

$$-\eta \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \frac{mgp}{k\mathrm{T}_0}\left(\frac{\mathrm{T}-\mathrm{T}_0}{\mathrm{T}}\right). \quad . \quad . \quad (13)$$

For convenience we make the transformation (using (2)):

$$\left.\begin{aligned} x &= \log\frac{r}{r_0}; \qquad x_m = \log\frac{r_m}{r_0} = \frac{\mathrm{T}_1 - \mathrm{T}_0}{2a(1+be^{-ay})}, \\ x_0 &= \log\frac{r_0}{r_0} = 0. \end{aligned}\right\} \quad . \quad (14)$$

Then (13) takes the simplified form :

$$-\eta\frac{\partial^2 v}{\partial x^2} = r_0^{\,2}\, e^{2x}\frac{mgp}{k\mathrm{T}_0}\left(\frac{\mathrm{T}-\mathrm{T}_0}{\mathrm{T}}\right); \quad . \quad . \quad . \quad (15)$$

and the temperature expression (1) becomes :

$$\mathrm{T} = \mathrm{T}_1 - ax(1+be^{-ay}). \quad . \quad . \quad . \quad . \quad (16)$$

By substituting this T in (15), we obtain a function of the variables x and y only. Now expand this as a power series in x, neglecting terms of higher than second order,

and integrate using Law I. to determine the constant of integration, that is, $\frac{\partial v}{\partial x}=0$, when $x=x_m$. We thereby obtain for the result of the first integration:

$$-\frac{\partial v}{\partial x}=\frac{pmgr_0^2}{k\eta T_0 T_1}\Big[(T_1-T_0)\{x-x_m+x^2-x_m^2+\tfrac{2}{3}(x^3-x_m^3)\}$$
$$-\frac{T_0}{T_1}a(1+be^{-ay})\left\{\frac{x^2-x_m^2}{2}+\tfrac{2}{3}(x^3-x_m^3)\right\}$$
$$-\frac{T_0a^2}{3T_1^2}(1+be^{-ay})^2(x^3-x_m^3)\Big],$$

which may be written

$$\Big(\text{since } a\{1+be^{-ay}\}=\frac{T_1-T_0}{2x_m} \text{ by (14)}\Big):$$

$$-\frac{\partial v}{\partial x}=\frac{pmgr_0^2(T_1-T_0)}{kn T_0T_1}\Big[x-x_m+x^2-x_m^2+\tfrac{2}{3}(x^3-x_m^3)$$
$$-\frac{T_0}{2x_mT_1}\left\{\frac{x^2-x_m^2}{2}+\tfrac{2}{3}(x^3-x_m^3)\right\}$$
$$-\frac{T_0(T_1-T_0)}{12T_1^2x_m^2}(x^3-x_m^3)\Big]. \quad . \quad (17)$$

Now integrate again and use the fact that $v=0$ when $r=r_0$, or by (14), $v=0$ when $x=0$. Rearranging terms in powers of x gives:

$$v=\frac{pmgr_0^2(T_1-T_0)}{k\eta T_0T_1}\left[Ax-\frac{x^2}{2}-Bx^3-Cx^4\right], \quad (18)$$

where

$$A=x_m\left(1-\frac{T_0}{3T_1}+\frac{T_0^2}{12T_1^2}\right)+x_m^2\left(1-\frac{T_0}{3T_1}\right)+\tfrac{2}{3}x_m^3;$$

$$B=\frac{1}{3}\left(1-\frac{T_0}{4T_1x_m}\right);$$

$$C=\frac{1}{6}\left(1-\frac{T_0}{2T_1x_m}+\frac{T_0(T_1-T_0)}{8T_1^2x_m^2}\right).$$

By Law I. the maximum convection velocity v_m occurs at the outer boundary of the film, where $x=x_m$. Substitute this in (18), and collect terms:

$$v_m=\frac{pmgr_0^2(T_1-T_0)x_m^2}{k\eta T_0T_1}$$
$$\left\{\frac{1}{2}-\frac{11T_0}{48T_1}+\frac{T_0^2}{16T_1^2}+x_m\left(\frac{2}{3}-\frac{T_0}{4T_1}\right)+\frac{x_m^2}{2}\right\},$$

or, if we replace x_m in the first factor by (14), we get:

$$v_m = \frac{pmgr_0^2(T_1-T_0)^3}{8k\eta T_0 T_1 a^2(1+be^{-ay})^2} \left\{1-\frac{11T^0}{24T_1}+\frac{T_0^2}{8T_1^2}+x_m\left(\frac{4}{3}-\frac{T_0}{2T_1}\right)+x_m^2\right\} = \frac{pmgr_0^2(T_1-T_0)^3N}{8k\eta T_0 T_1 a^2(1+be^{-ay})^2}, \quad . \quad . \quad . \quad (19)$$

where

$$N=1-\frac{11\rho}{24}+\frac{\rho^2}{8}+x_m\left(\frac{4}{3}-\frac{\rho}{2}\right)+x_m^2$$

and

$$\rho=\frac{T_0}{T_1}.$$

7. *Heat Transfer.*

The heat transfer per unit area per second (with the help of (1)) is:

$$q=-K\left(\frac{\partial T}{\partial r}\right)_{r=r_0} \begin{cases} q=h(T_1-T_0), \\ q=K\dfrac{a}{r_0}(1+be^{-ay}) \end{cases}, \quad . \quad (20)$$

where K is the conductivity coefficient and h is the heat transfer coefficient. The total heat transferred from a cylinder of height L is:

$$Q=\int_0^L 2\pi r_0 q\,dy = 2\pi aK\int_0^L (1+be^{-ay})dy = 2\pi aK\left\{L+\frac{b}{\alpha}(1-e^{-aL})\right\}. \quad (21)$$

We now use the approximate empirical relation:

$$L=\frac{b}{\alpha}; \qquad b=\alpha L, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

which is evidently equivalent, when the exponential term is neglected, to the more accurate:

$$\frac{Q}{2}=2\pi aKL=2\pi aK\frac{b}{\alpha}(1-e^{-aL}). \quad . \quad . \quad (23)$$

This relation (23) has been shown * to be accurately true in case of tall vertical cylinders for a certain critical height

* Kimball and King, "Theory of Heat Conduction and Convection from Tall Hot Vertical Cylinders and High Walls at Uniform Temperature." [This paper will be published in an early issue of this Magazine.—Eds.]

$L = H$ (about 40 cm. for air) which separates the stream-line state of affairs below from the turbulent state above it. The present theory indicates that also for short vertical surfaces (lower than 40 cm.) and having no upper turbulent region, this relation is more accurate, using $L = H = 40$ cm. (see Tables IV. and V. below), and seems to indicate that it is independent of L, the cylinder height, whether this be greater or less than the critical H.

We may interpret (23) which is equivalent to (22) in connexion with Law II. which postulates that half the heat is carried across the film by pure conduction, and half convected up away inside the film. Thus the middle term of (23) may represent pure conduction across the film, and the right member of (23) the heat convected up and away inside the outer boundary of the film at the top. In point of fact, the right member is the excess supply (by conduction) of heat in the lower regions due to excess temperature gradient down there. Any inaccuracy in the results from this experimental relation (22) enters to the one-fourth power (see eq. (27) below), and so is about 2 per cent. for as much as 8 per cent. error involved in (22).

We now proceed to calculate the heat convected up and away vertically inside the outer boundary of the film at the top. By elementary kinetic theory, $\frac{3}{2}nk\mathrm{T}$ is the energy per unit volume of gas at temperature T. The excess energy per unit volume that comes from the hot cylinder is:

$$\tfrac{3}{2}nk(\mathrm{T}-\mathrm{T}_0),$$

and the flux, or heat transported away vertically through unit area in unit time is the above multiplied by the velocity:

$$q = \tfrac{3}{2}nk(\mathrm{T}-\mathrm{T}_0)v. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The heat convected away inside the film $\left(\text{which by Law II. is } \frac{\mathrm{Q}}{2}\right)$ is the integral of this flux (24) over the total surface through which heat flows, which in this case extends from the hot cylinder at the top, out to the outer boundary of the film, where v is maximum, and where also $r^m - r_\mathrm{L}$ and $x_m = x_\mathrm{L}$ as shown in Section IX. below:

$$\frac{\mathrm{Q}}{2} = 2\pi \int_{r_0}^{r_\mathrm{L}} qrdr = 3\pi k \int_{r_0}^{r_\mathrm{L}} n(\mathrm{T}-\mathrm{T}_0)vrdr,$$

or, using (12) and (14):

$$\frac{\mathrm{Q}}{2} = 3\pi p \int_{r_0}^{r_\mathrm{L}} \frac{(\mathrm{T}-\mathrm{T}_0)}{\mathrm{T}} vrdr = 3\pi p r_0^2 \int_0^x \frac{\mathrm{T}-\mathrm{T}_0}{\mathrm{T}} e^{2x} v dx. \quad (25)$$

The integration of (25) involves expanding e^{2x} and $\frac{T-T_0}{T}$ in powers of x (using $T=T_1-\frac{x(T_1-T_0)}{2x_L}$ according to (14) and (16)). The product of these two series must be multipled by (18), and we retain no terms higher than the third power before integration. Then, after integration and collection of terms in powers of $\frac{T_0}{T_1}$ and x_L, we have:

$$\frac{Q}{2}=\frac{\pi p^2 mg r_0^4 (T_1-T_0)^2 x_L^3}{k\eta T_0 T_1^2}\,M, \quad . \quad . \quad . \quad (26)$$

where

$$M=A'+x_L B'+x_L^2 C'+x_L^3 D'+x_L^4,$$

where

$$A'=1-\frac{15}{16}\rho+\frac{13}{24}\rho^2-\frac{23}{192}\rho^3+\frac{\rho^4}{64},$$

$$B'=\frac{5}{2}-\frac{125}{48}\rho+\frac{5}{6}\rho^2-\frac{\rho^3}{8},$$

$$C'=\frac{9}{2}-\frac{19}{8}\rho+\frac{\rho^2}{2}$$

$$D'=\frac{17}{6}-\rho, \quad \text{and} \quad \rho=\frac{T_0}{T_1}.$$

Now replace x_L^3 in the first factor of (26) by

$$x_L=x_m=\frac{T_1-T_0}{2a},$$

at the top where e^{-aL} is negligible according to (14), and set (26) equal to (23), and solve for a:

$$\left.\begin{aligned} a&=\left(\frac{p^2 mg\, r_0^4 (T_1-T_0)^5 M}{16 k\eta T_0 T_1^2 KL}\right)^{\frac{1}{4}},\\ \frac{a}{r_0}&=\frac{T_1-T_0}{2}\left(\frac{p^2 mg (T_1-T_0) M}{k\eta T_0 T_1^2 KL}\right)^{\frac{1}{4}}. \end{aligned}\right\} \quad . \quad . \quad . \quad (27)$$

8. *Experimental Checks.*

1. We obtain the well-known five-fourths power law for variation of heat transfer with temperature excess when (27) is substituted in (23):

$$Q=2\pi r_0 (T_1-T_0)\left(\frac{p^2 mg (T_1-T_0) M K^3 L^3)}{k\eta T_0 T_1^2}\right)^{\frac{1}{4}}. \quad (28)$$

2. From (20) and (27) we have the heat transfer coefficient:

$$h = \frac{(1+be^{-ay})}{2}\left(\frac{p^2 mg(T_1 - T_0)MK^3}{k\eta T_0 T_1^2 L}\right)^{\frac{1}{4}}. \quad . \quad (29)$$

This expression also seems to check known data, being proportional to the square root of the pressure, the 4th root of the temperature excess, and the 3/4 power of the heat conductivity coefficient.

3. When (27) is substituted in (19) we get the maximum value of v at the height y:

$$v_m = \frac{N}{2(1+be^{-ay})^2}\sqrt{mg\frac{(T_1-T_0)KL}{k\eta T_0 M}}. \quad . \quad (30)$$

For the maximum at the top of the cylinder, where $y=L$ and the exponential term is negligible, we have:

$$v_m = \frac{N}{2}\sqrt{\frac{mg(T_1-T_0)KL}{k\eta T_0 M}}. \quad . \quad . \quad . \quad (31)$$

This expression shows that v_m varies as the square root of the temperature excess, which checks Schmidt's * data for a hot plate and is probably true for cylinders. It is also independent of the pressure.

4. The constant a given by (27) is dimensionally a temperature, as is readily shown by replacing all of the physical constants in (27) by their dimensional equivalents, remembering that M is of zero dimensions. All the formulæ in which "a" enters are then easily shown to be of correct dimensions.

9. *The Film Thickness.*

From (2) we have:

$$x_m = \log\frac{r_m}{r_0} = \frac{T_1-T_0}{2a(1+be^{-ay})}, \quad . \quad . \quad . \quad (32)$$

and by expanding $\exp\frac{T_1-T_0}{2a\{1+be^{-ay}\}}$, we have to first approximation:

$$r_m - r_0 = \frac{r_0(T_1-T_0)}{2a\{1+be^{-ay}\}}, \quad . \quad . \quad . \quad . \quad (33)$$

for the thickness of the isothermal film.

If we substitute (23) in (5), we have

$$x_L = \log\frac{r_L}{r_0} = \frac{T_1-T_0}{2a}, \quad . \quad . \quad . \quad . \quad (34)$$

* E. Schmidt, *Zeits. des Gesamte Kalte Industrie*, p. 213 (Nov. 1928).

and to first approximation :

$$r_L - r_0 = \frac{r_0(T_1 - T_0)}{2a}, \quad . \quad . \quad . \quad . \quad (35)$$

for the thickness of the Langmuir film.

Substituting (27) in (33) and (35) gives :

$$r_m - r_0 = \left(\frac{k\eta T_0 T_1^2 KL}{p^2 mg(T_1 - T_0)M}\right)^{\frac{1}{4}} \frac{1}{\{1 + be^{-ay}\}} \quad . \quad (36)$$

and

$$r_L - r_0 = \left(\frac{k\eta T_0 T_1^2 KL}{p^2 mg(T_1 - T_0)M}\right)^{\frac{1}{4}}, \quad . \quad . \quad . \quad (37)$$

for first approximate expressions for the thickness of the isothermal film and the Langmuir film. These are seen to vary inversely as the fourth root of $T_1 - T_0$ which is in accord with known data. Also note that the isothermal film approaches the Langmuir film asymptotically as y increases, as pointed out in section three above (see fig. 1).

10. *Calculation of Heat Transfer Coefficients.*

From (20) we have:

$$h = \frac{Ka(1 + be^{-ay})}{r_0(T_1 - T_0)} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

If we eliminate $a(1 + be^{-ay})$ from (38) by means of (32), we have :

$$h = \frac{K}{2r_0 x_m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and since $x_m = x_L/(1 + be^{-ay})$ by (32) and (34), we have :

$$h = \frac{K(1 + be^{-ay})}{2r_0 x_L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Comparison of (29) and (40) shows that

$$r_0 x_L = \left(\frac{k\eta T_0 T_1^2 KL}{p^2 mg(T_1 - T_0)M}\right)^{\frac{1}{4}} = \left(\frac{G}{M}\right)^{\frac{1}{4}} \quad . \quad . \quad . \quad (41)$$

This formula is useful in calculating heat transfer coefficients. It is written in this form because

$$G = \frac{k\eta T_0 T_1^2 KL}{p^2 mg(T_1 - T_0)}$$

is independent of r_0, whereas M depends on $x_L = \log \frac{r_L}{r_0}$ and hence on r_0 also.

Using C.G.S. units we have calculated G for a series of temperature excesses over 20° C. = 293°, taking L = 40.

Thus,

$k = 1{\cdot}37 \times 10^{-16}$ = Boltzmann's constant,
$m = 47 \times 10^{-24}$ = mass of an air molecule,
$p = 1{\cdot}013 \times 10^{6}$ = pressure in dynes per cm.^{2}
$g = 980$ = acceleration of gravity,
$T_0 = 293°$.

The other physical constants, K the conductivity coefficient. η the viscosity coefficient, and T_1 will vary for different temperatures. Using C.G.S. units, we get:

TABLE I.

$T_1 - T_0$.	10°.	20°.	30°.	40°.	50°.	60°.	
G.....	·1366	·07564	·05502	·04541	·03978	·03589	
$T_1 - T_0$.	70°.	80°.	90°.	100°.	110°.	125°.	140°.
G.....	·03367	·03202	·03023	·02897	·02858	·02803	·02777

The computation of the M's is not so easy. We assume various values of x_L from ·6 to ·02, and calculate M in each case and tabulate for a series of values of $\frac{T_0}{T_1} = \rho$, the temperature ratio. Then with the help of (40) and (41) we can get the corresponding h and r_0. For larger values of x_L than ·6 the series involved in the expansions converges slowly if at all. This corresponds to a cylinder diameter of about 1 cm., which is hence the lower limit of the applicability of this theory. For $x_L = {\cdot}02$ the cylinder diameters are about 50 cm. Larger cylinders have not been considered.

Griffiths and Davis's Report* gives heat transfer coefficients for cylinders of radius 8·7 cm. It turns out that the various values of M for this radius and the indicated temperature excesses are:

TABLE II.

$T_1 - T_0$.	10°.	20°.	30°.	40°.	50°.	60°.	
M.....	·570	·572	·575	·583	·589	·595	
x_L.....	·079	·069	·066	·062	·059	·058	
$T_1 - T_0$.	70°.	80°.	90°.	100°.	110°.	125°.	140°.
M.....	·600	·605	·610	·620	·627	·638	·650
x_L.....	·057	·055	·054	·054	·053	·053	·052

* *Loc. cit.* Table 8 A.

This table is taken from a larger table * giving values of M for the indicated range of temperature excess for a whole series of x_L's in each case. The present table above includes only the cases where $r_0 = 8{\cdot}7$ according to (41), for which we have experimental data.

Eqs. (40) and (41) enable us with the help of the above tables to calculate h, remembering that the conductivity coefficient (for calories) is $K = 568\sqrt{\frac{T_1}{273}} \times 10^{-7}$:

TABLE III.

Theoretical Heat Transfer Coefficients h for Cylinders of Radius 8·7 cm., and Height 40 cm. in calories per cm ^{2} per degree per sec.

$T_1 - T_0$.	10°.	20°.	30°.
$h \times 10^5$	$4{\cdot}26(1+6e^{-{\cdot}15y})$	$5{\cdot}04(1+6e^{-{\cdot}15y})$	$5{\cdot}51(1+6e^{-{\cdot}15y})$
$T_1 - T_0$.	40°.	50°.	60°.
$h \times 10^5$	$5{\cdot}92(1+6e^{-{\cdot}15y})$	$6{\cdot}24(1+6e^{-{\cdot}15y})$	$6{\cdot}50(1+6e^{-{\cdot}15y})$
$T_1 - T_0$.	70°.	80°.	90°.
$h \times 10^5$	$6{\cdot}74(1+6e^{-{\cdot}15y})$	$6{\cdot}99(1+6e^{-{\cdot}15y})$	$7{\cdot}16(1+6e^{-{\cdot}15y})$
$T_1 - T_0$.	100°.	110°.	125°.
$h \times 10^5$	$7{\cdot}34(1+6e^{-{\cdot}15y})$	$7{\cdot}47(1+6e^{-{\cdot}15y})$	$7{\cdot}7(1+6e^{-{\cdot}15y})$
$T_1 - T_0$.		140°.	
$h \times 10^5$		$7{\cdot}9(1+6e^{-{\cdot}15y})$	

In Table III. we have chosen $b = 6$ and $\alpha = {\cdot}15$, being related by (22) taking L = 40, since a careful study of Griffiths and Davis' Table 8 A and fig. 15 indicates that this choice of the empirical constant b seems to fit the experimental data. These authors give the *average* value of the heat transfer for cylinders of various heights :

* Kimball and King, *loc. cit.*

$$\bar{q} = (T_1 - T_0)\bar{h} = \frac{T_1 - T_0}{L}\int_0^L h\,dy$$

$$= \frac{T_1 - T_0}{L}\int_0^L \frac{Ka}{(T_1 - T_0)r_0}(1 + be^{-ay})\,dy$$

$$= \frac{Ka}{Lr_0}\left(L + \frac{b}{a}(1 - e^{-aL})\right) = \frac{2Ka}{r_0}. \qquad (42)$$

The last line follows from (22) and (23). In Table III., however, the factors in front of the parenthesis are seen by (38) to be $\frac{Ka}{r_0(T_1 - T_0)}$. To obtain $\bar{q}$ we have only to multiply these by $2(T_1 - T_0)\sqrt[4]{\frac{40}{L}}$, where the radical factor introduces the L corresponding to the height considered instead of the L = 40 used in Table III. Thus for 10° we have:

$$\bar{q} \times 10^5 = 4{\cdot}26(T_1 - T_0) \cdot 2\sqrt[4]{\frac{40}{L}} = (85{\cdot}2)\sqrt[4]{\frac{40}{L}}.$$

The theoretical heat transfer, as shown in Table IV., is smaller than the experimental results in much the same proportion as given by the previous * vertical hot plate study, and presumably for the same reason, *i. e.*, neglect of inertia effects in the force equation (11) above. This interpretation of the discrepancy between theory and experiment is confirmed by the fact that the discrepancy in question shown in Table IV. is much greater for the shortest cylinder, where it is to be expected that the effect of inertia would be greatest.

An alternative check-up on the experimental data may be made by a different use of (22) and (23), shown †, in case of high walls and tall cylinders, to be accurate when L is replaced by H = 40 the critical height, being thus independent of L for tall surfaces. Likewise for short surfaces we may treat (22) and (23) as independent of L, and assume that these formulæ hold only when L is replaced by H = 40. In this case the constant "a" of (42) is independent of L, and is the one used in Table III. We also use the b and a of Table III., and since now $\frac{b}{a} = H = 40$

* Kimball and King, *loc. cit.*
† Kimball and King, *loc. cit.*

and e^{-aL} is no longer negligible, we have instead of (42) :

$$q = \frac{Ka}{Lr_0}(L + 40(1 - e^{-aL})). \quad . \quad . \quad . \quad (43)$$

TABLE IV.

Mean Heat Transfer, $q \times 10^2$ for Cylinders of Radius 8·7 cm. and four different Heights, compared with Experimental Data by Griffiths and Davis, Table 8 A, using $b = \alpha L$.

Height in cm.		10°.	20°.	30°.	40°.	50°.
28·8	Th.	·092	·218	·357	·512	·674
	Ex.	·100	·238	·393	·562	·743
15·2	Th.	·1085	·257	·421	·603	·795
	Ex.	·152	·342	·548	·773	1·01
8·00	Th.	·128	·301	·494	·708	·933
	Ex.	·218	·485	·780	1·09	1·41
4·65	Th.	·146	·345	·566	·811	1·07
	Ex.	·270	·625	1·025	1·45	1·91

Height in cm.		60°.	70°.	80°.	100°.
28·8	Th.	·843	1·02	1·21	1·59
	Ex.	·933	1·13	1·35	1·79
15·2	Th.	·994	1·20	1·43	1·87
	Ex.	1·24	1·48	1·73	2·26
8·00	Th.	1·17	1·41	1·68	2·20
	Ex.	1·75	2·10	2·47	3·21
4·65	Th.	1·34	1·62	1·92	2·52
	Ex.	2·37	2·87	3·38	4·44

Thus for $T_1 - T_0 = 10°$ and $L = 15{\cdot}2$ cm., using the h of Table III., we have :

$$q \times 10^5 = \frac{10(4{\cdot}26)}{15{\cdot}2}\int_0^{15{\cdot}2}(1 + 6e^{-\cdot 15y})dy$$

$$= \frac{42{\cdot}6}{15{\cdot}2}(15{\cdot}2 + 40(1 - e^{-2{\cdot}26})) = 143{\cdot}2. \quad . \quad (44)$$

Thus Table V. shows $\bar{q} \times 10^2$ and how theory checks with experiment (Table 8 A, Griffiths and Davis) using in this case for (22), $b = \alpha H = 40\alpha$, independent of L instead of $b = \alpha L$ as in Table IV. For very short cylinders (4·65 cm.) the

TABLE V.

Comparison of Theoretical and Experimental Values of $\bar{q} \times 10^2$ in calories per cm.^{2} per sec., using $b = 40\alpha$.

Height in cm.		10°.	20°.	30°.	40°.	50°.
28·8	Th.	·101	·239	·392	·561	·739
	Ex.	·100	·238	·393	·562	·743
15·2	Th.	·1432	·339	·556	·797	1·049
	Ex.	·152	·342	·548	·773	1·01
8·00	Th.	·1914	·453	·743	1·065	1·402
	Ex.	·218	·485	·780	1·09	1·41
4·65	Th.	·2266	·537	·879	1·259	1·66
	Ex.	·270	·625	1·025	1·45	1·91

Height in cm.		60°.	70°.	80°.	100°.
28·8	Th.	·925	1·118	1·325	1·74
	Ex.	·933	1·13	1·35	1·79
15·2	Th.	1·312	1·587	1·879	2·469
	Ex.	1·24	1·48	1·73	2·26
8·00	Th.	1·753	2·122	2·514	3·30
	Ex.	1·75	2·10	2·47	3·21
4·65	Th.	2·077	2·512	2·977	3·91
	Ex.	2·37	2·87	3·38	4·44

theoretical results are too small. Inertia is probably a disturbing factor in this region. For taller cylinders, however, ranging up to the critical height H = 40 cm., the present theory affords an excellent check, and is probably within the limits of experimental error. The theory, moreover, is flexible, owing to the empirical relation $b = 40\alpha$, and hence for different values of b, gives different sets of theoretical data. Thus a smaller b than 6, used in Table V.,

would give smaller values of the theoretical heat transfer $\bar{q}$, and might fit the facts better in case inertia effects were included in the analysis.

The fact that Table V. affords a check on experiment which is so much better than Table IV. is perhaps best interpreted as experimental evidence that the relation (22) or (23) is independent of height for short hot vertical surfaces as well as tall ones, and holds only for the critical height H, since $b = \alpha H = 40\alpha$ is independent of L as used in computing the data of Table V.

11. *Conclusions.*

1. For short hot vertical cylinders, stream-line conditions prevail, and the mechanism of heat transfer is pure conduction through gas in stream-line motion, as was the case for low vertical planes.

2. The empirical laws applicable to low vertical hot planes, deduced from the data of Schmidt and others, probably also hold for short hot vertical cylinders since the present theory based on these laws affords very satisfactory experimental checks :

Law I. The locus of the maxima of the convection velocity curves is an isothermal surface (the film boundary) at temperature half-way between the hot surface and the ambient air.

Law II. Half the heat supply is convected up away inside this surface and half outside it.

Further experimental investigation of velocity and temperature distribution, especially within the film, is desirable.

3. The heat transfer from the hot cylinders is proportional to the five-fourths power of the temperature excess.

4. The maximum convection velocity is (*a*) proportional to the square root of the temperature excess, (*b*) independent of pressure.

5. The film thickness varies inversely as the fourth root of the temperature excess, and is notably smaller for cylinders of less than 6 cm. diameter. The isothermal film thickness approaches that of the Langmuir film at the top of the cylinder.

6. Heat transfer coefficients are proportional (*a*) to the square root of the pressure, (*b*) to the fourth root of the temperature excess, and increase markedly for cylinders of less than 6 cm. diameter.

7. The theory herein is accurate and useful for cylinders with diameter as small as one centimetre. Smaller cylinders can be treated by considering additional terms in the expansions.

8. The experimental data of Griffiths and Davis on heat transfer from cylinders under 40 cm. high and 17·4 cm. diameter is checked to a satisfactory degree of accuracy.

9. The experimental data seems to indicate that the relation $H = \frac{b}{\alpha}(1-e^{-\alpha H})$, involving the critical height H=40 cm. separating stream-line conditions below from turbulence above, is independent of L, the cylinder height, for short cylinders as well as tall ones.

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XXXVIII. *The Geissler Discharge in Argon, a re-analysis of some Collector Characteristics**. *By* K. G. Emeléus, *M.A., Ph.D., and* R. H. Sloane, *M.Sc.*†, *Department of Physics, The Queen's University of Belfast.*

1. *Introduction.*

THE cold cathode glow discharge in argon between disk electrodes was investigated some years ago by Emeléus and Harris ‡, when it was shown by the application of one of Langmuir's probe methods that the processes responsible for the passage of the current in parts now referred to as of *plasma* type § were similar to those effective in some other forms of discharge which had been studied in the same way. Following upon this, Emeléus and Brown ‖, as a result of an investigation of the characteristic curves of collectors at a negative potential, came to the conclusion that there was also present in the negative glow in argon, and other gases, a group of electrons with an average energy of the order of 30 electron volts, and a concentration of from 0·01 to 0·001 of that of the slower electrons. The presence of fast electrons

* Communicated by the Authors.
† Musgrave Student.
‡ Emeléus and Harris, Phil. Mag. iv. p. 49 (1927).
§ Tonks and Langmuir, Phys. Rev. xxxiv. p. 876 (1930).
‖ Emeléus and Brown, Phil. Mag. vii. p. 17 (1929).

has been confirmed in several spectroscopic investigations *, and characteristics similar to those of Emeléus and Brown have been found in similar parts of discharges at higher pressures by Wehrli and Bächtiger †. Some new ways for studying the cathode dark space have also been developed ‡, and have yielded information for argon which supplements that obtained by Langmuir's method for the plasma.

It is now known that the use of collectors in discharges such as these is open to certain objections. Apart from the question of distortion of the discharge, which we shall not go into here, it has been found that secondary effects may occur at the surface of the collector and in the sheath. These are of special importance when there is a large concentration of metastable atoms in the discharge, or a large density of diffusing radiation of short wave-length. There is some dispute about the relative importance of metastable atoms and radiation, but it is definitely established that they may, together with secondary emission from the collector due to electrons and positive ions, profoundly modify the characteristic curves §.

Very recently Seeliger and Hirchert have published the results of an investigation of the positive column in the noble gases, which have also shown that the electron currents recorded may be modified both by certain impurities in the discharge, and by the length of the collector used, if this is a wire ‖. Three types of curves relating the logarithm of the electron current to the retarding potential were distinguished. The first is the single straight line of Langmuir's original theory; in the second, or K type, two lines occur in such a way that they can be referred to the presence of two slow-speed Maxwellian distributions; the third type, which is S-shaped, is perhaps associated with the liberation of some impurity from the walls of the discharge-tube, and has also been found by Wehrli and Bächtiger ¶, and in the low-voltage arc **. Seeliger and

* K. G. Emeléus and F. M. Emeléus, Phil. Mag. viii. p. 383 (1929); Druyvesteyn, *Zeit. f. Phys.* lxii. p. 764 (1930); Druyvesteyn and Warmoltz, *Zeit. f. Phys.* lxviii. p. 378 (1931).

† Wehrli and Bächtiger, *Helv. Phys. Acta*, iv. p. 290 (1931).

‡ Carmichael and Emeléus, Phil. Mag. viii. p. 909 (1929); Brown and Thomson, Phil. Mag. viii. p. 918 (1929); Emeléus and Beck, Phil. Mag. xi. p. 54 (1931).

§ *Cf.* Oliphant, Roy. Soc. Proc. cxxxii. A, p. 631 (1931); Langmuir and Found, Phys. Rev., Jan. 15 (1932).

‖ Seeliger and Hirchert, *Ann. d. Phys.* xi. p. 817 (1931).

¶ *Loc. cit.*

** Druyvesteyn, *Zeit. f. Phys.* lxiv. p. 781 (1930).

Hirchert have refrained from analysing the K type for two electron groups, since it degenerates into a single line if the length of the collector is decreased. This change evidently calls for further investigation, but it has been shown that the two groups in a low-voltage arc in argon account well for the energy delivered to a collector, measured thermally *, and we shall assume meanwhile that two groups exist in the discharge independently of the presence of the collector where we have found characteristics of the K type. To this an exception must naturally be made if the slowest group can be associated with secondary emission (§ 2, iii.).

During the last few years the glow discharge in helium has been under investigation here by a combination of probe and optical methods. Incidental to this the validity of the straightforward interpretation of the collector characteristics under some of the conditions of the first study of argon has been carefully examined, and conclusions which supplement those of Seeliger and Hirchert have been reached, specially applicable to the boundary between the negative glow and cathode dark space. A detailed account of these will be published shortly. The present paper is a revision of part of the earlier work on argon which is made necessary by this and by our increased knowledge of secondary emission, together with an observation of Wehrli and Bächtiger † on the random current of electrons. In general, it must be recognized that the information afforded by analyses of collector characteristics in the noble gases at low current density, particularly for the negative sections of the discharge, is less exact than when the method is applied to mercury arcs at much higher current density.

2. *Revision of Earlier Conclusions.*

The new work referred to in § 1 makes it necessary to reconsider several points in the work of Emeléus and Harris, and of Emeléus and Brown. Some of these are more conveniently dealt with in conjunction with our results for helium, but three will be gone into here. The first, and most important point is that there is actually no evidence for the presence of a fast directed group of electrons at the boundary between the negative glow and cathode dark space. The second is that the random current density of electrons in the discharge at the highest pressure analysed by Emeléus and Harris is too low, and the third that it is desirable to discuss

* van Voorhis, Phys. Rev. xxx. p. 318 (1927).
† *Loc. cit.*

in some detail whether the collector characteristics afford evidence for the presence of an additional group of slow electrons in the discharge at 0·29 mm. described in the same paper.

(i.) In the first paper, it was concluded that the boundary region between the negative glow and cathode dark space contained a large reversed field, whilst in the second paper this was correlated with a form of characteristic curve which was interpreted as showing that a fast group of directed electrons, not resolvable from the positive ions, was there passing into the negative glow. An essential feature of the characteristic curves is that the positive ion line extrapolates to an electron current at the space potential. This in itself is probably not sufficient to make it necessary to reject the data for the electrons and the space-potential found from the difference between the extrapolated positive ion current and the observed current, a somewhat similar difficulty having been encountered by Langmuir *. The cathode dark space is, however, associated with an exaggerated form, in which the characteristic takes an S-shape, crossing the axis of voltage at a potential considerably negative to that of the negative glow, remaining concave to the axis of voltage until the potential of the negative glow is approached, and finally rising almost linearly. Fig. 1 shows an example of this, which has been found now to be due to the fact that the collector was in a region of positive space-charge, this form of curve being obtained when the characteristic curve of one collector is taken within the positive space-charge formed on another held at a negative potential. In our earlier work, for positions close to the negative glow, the intermediate concave part was supposed due to a directed stream of electrons, and the subsequent increase in electron current to the collector was analyzed in the usual way for electron temperature and concentration, and space-potential, leading to the apparent large reversed field. This procedure, and the conclusions based upon it, must be regarded as incorrect.

Whilst there is no difficulty in recognizing this type of curve when the form is so pronounced as in fig. 1, it is a matter of considerable difficulty to decide if there is distortion of the characteristics in the extreme negative boundary of the negative glow, where one side of the collector is exposed to the positive space-charge of the cathode dark space and the other to the negative glow plasma. It is suggested that a good working rule is, in the absence of

* Tonks and Langmuir, *loc. cit.*

satisfactory evidence to the contrary, to reject all data obtained by application of Langmuir's method of analysis for cold collectors to the cathode side of the place where the electron concentration is found to be a maximum, in about the brightest part of the negative glow. This rules out the improbably large reversed fields in the boundary, but also

Fig. 1.

Graph of ratio of collector current to tube current against potential of collector in volts negative to the anode, for collector in cathode dark space. Tube voltage 220; tube current 0·5 milliamp.; pressure 0·4 mm.; visual thickness of cathode dark space 0·54 cm.; collector 1 mm. into cathode dark space from negative glow; space potential in negative glow 18 volts negative to anode; the unit of the current ratio is $1{\cdot}15 \cdot 10^{-1}$ amp./amp. (Emeléus and Harris, run 90.) A positive ratio is a net current of electrons to the collector from the gas.

makes it necessary to seek another origin than that suggested by N. M. Carmichael for the dark space found there in oxygen *. The data for 0·9 mm. pressure (Emeléus and Harris, Table II.) must be rejected completely, for another reason (ii.). An approximate method for finding the electron concentrations in the boundary region is described in § 3.

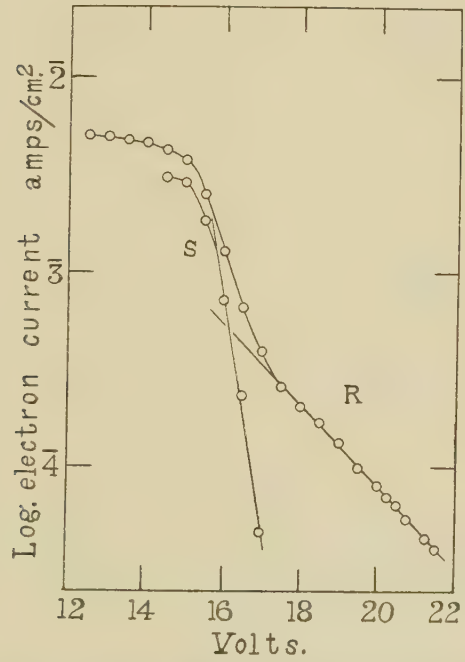
(ii.) Wehrli and Bächtiger † have pointed out that in the discharge at 0·9 mm. described in the first paper (Table II.) the random current density integrated over the cross-section of the tube is less than the current recorded as passing by an external galvanometer. On referring to the original data for these runs, we find that, as Wehrli and Bächtiger suggested, the semi-logarithmic curves are probably of Seeliger and Hirchert's S type. The rather negative potential of the transition to concavity to the potential axis has been taken to be the space-potential. Since there was an anode glow, but no positive column, the true space-potential was probably about 16 volts negative to the anode, but the characteristic curves were not followed so far. From the trend of the curves it appears probable that this explanation would allow an ample margin to remove the discrepancy between the tube current density and the random current density. There is, however, no indication from the original notes on these runs that the impurity to which Seeliger and Hirchert attribute this form of curve, should have been especially prominent in this instance, except that the relatively high current density may have caused liberation of this from the walls. It is also unlikely that it was associated with the pressure being higher than in the other cases, as although Wehrli and Bächtiger find this form of curve frequently in their work, the pressures employed by them are usually at least twenty times greater.

(iii.) The data of Table IV. and fig. 6 in the paper by Emeléus and Harris, for a discharge at a pressure of 0·29 mm. should possibly be modified to include a slower group of electrons in addition to the group already recorded. The semi-logarithmic curves are of Seeliger and Hirchert's K type, but the rise in electron current to the collector near the space-potential had previously been taken to be due to the retention of slow electrons released from the surface by some secondary process and included in the positive current to the collector for retarding potentials for electrons. The curve

* Carmichael, Phil. Mag. viii. p. 362 (1929).
† *Loc. cit.*

for the position nearest the anode, in which the effect is most marked, is shown in fig. 2. In the first paper, a break near R had been taken to be at the space-potential; we have now to consider whether it should not have been taken at S. If the space-potential is at the latter position, the rise between R and S shows that there are slow electrons present, the semi-logarithmic curve for which is shown as the separate line in the figure. Rather few experimental

Fig. 2.



Semi-logarithmic current-voltage characteristic curve for collector in the Faraday dark space; the abscissæ are potentials negative to the anode.

points are available, but the general course of the curves is unambiguous. Table I. contains the electron concentrations, and the new space-potential if we assume that there are two electron groups present. Table II. is obtained by assuming that there is only one group; it differs from Table IV. of the paper by Emeléus and Harris in that the random current at the space-potential has been taken to be that corresponding to the point S, instead of, as previously, that corresponding

to R. We are indebted to Mr. E. W. Pike for calling our attention to the necessity for this alteration. A difficulty

TABLE I.

Alternative analyses for discharge at 0·29 pressure.

Position in discharge.	Faraday dark space.		Negative glow.			
Distance from anode: cm.*	3·17	3·56	3·79	3·98	4·06	4·17
Space potential: volts $-^{VE}$ to anode	15·75	15·95	15·35	15·5	14·90	14·45
Electron concentrations: c.c. (i.) 10^7	16·0	15·6	17·6	12·1	12·3	7·0
Electron concentrations: c.c. (ii.) 10^8	8·35	4·24	3·60	3·53	2·64	1·64
Electron temperatures: volts. (i.)	3·10	3·92	3·44	3·71	3·92	4·46
Electron temperatures: volts. (ii.)	0·55	0·70	0·79	0·48	0·62	0·54
Concentration of metastable atoms: 10^{11}/c.c. †	6·5	3·7	3·4	2·6	2·2	1·3

* The 4·28 cm. position has been rejected (§2, i.); the two next positions, although giving rather low electron concentrations, yielded good characteristics and are retained.

† Alternative to slow group of electrons.

TABLE II.

Analysis of discharge in argon: single group of electrons; pressure 0·29 mm.; tube volts 275; tube-current density 0·03 milliamp./cm.^{2}; cathode dark space 0·72 cm. thick*; anode to cathode 5 cm.

Distance from anode: cm.	3·17	3·56	3·79	3·98	4·06	4·17
Space potential: volts $-^{VE}$ to anode	18·3	17·5	16·8	16·6	16·1	15·7
Electron concentration: 10^8/c.c.	4·50	2·94	2·97	2·36	2·33	1·36
Electron temperature: volts.	3·10	3·92	3·44	3·71	3·92	4·46

* By visual observation.

that arises in both cases is that the boundary of the cathode dark space, judged visually, was 4·3 cm. from the anode,

and the brightest part of the negative glow 4 cm. from the anode. This means that the concentration of electrons in either case is not a maximum at the brightest part of the negative glow, but somewhere in the Faraday dark space, which is contrary to what is usually found.

On the original interpretation it is possible to estimate the concentration of metastable atoms in the discharge if it is assumed, (*a*) that the whole of the secondary emission is due to these, and (*b*) that each metastable atom reaching the collector sets free one electron. It is unlikely that any other agent could cause an emission of this magnitude. An efficiency of unity is in accord with some theoretical work by Massey on the interaction between excited atoms and metal surfaces*. The concentration of metastable atoms is then obtained from the concentration of slow electrons on the alternative analysis by multiplying by the square root of the ratio of the mass of an argon atom to that of an electron and by the square root of the inverse ratio of temperatures. The temperature of the argon atoms has been assumed to be 500° K. Strictly speaking, the electron emission from the probe should not be taken as the current of slow electrons to the collector if these are real, but as the difference between the total current to the probe at the space-potential if the slow electrons are not actually present in the plasma, and the total current somewhere in the neighbourhood of the space-potential on the alternative assumption. The method adopted is sufficiently precise, as the other quantities involved, especially the temperature of the argon metastable atoms, cannot be found exactly. The results obtained in this way are also shown in Table I. The increase in concentration of metastable atoms on passing into the Faraday dark space is in accord with the occurrence of "aureoles" to the positive side of the main negative glow †, and, probably, with the behaviour of the helium band spectrum in the glow discharge in this gas ‡. On the other hand, the magnitude of the currents to which they give rise is surprisingly large compared with the currents of electrons and ions from the plasma to the collector. No such effect appears, for example, with the collector characteristic which has been published by Oliphant for a positive column in neon §.

It appears to be impossible, with the information available, to decide definitely between the two interpretations of the

* Massey, Proc. Camb. Phil. Soc. xxvi. p. 386 (1930).
† Druyvesteyn, *Zeit. f. Phys.* lvii. p. 292 (1929).
‡ Brown, Cowan, and Emeléus, 'Nature,' April 18 (1931).
§ Oliphant. *loc. cit.*

characteristic curves, but that which invokes metastable atoms, or some other source of secondary emission, is more nearly in accord with the view that the general course of the concentration of electrons in discharges such as these is shown by the change in brightness of the discharge from point to point. With regard to the data for the other runs described by Emeléus and Harris, these are considered satisfactory as they stand with the changes indicated in (i.) and (ii.); but it is questionable if it can be regarded as definitely proved that similar considerations to those advanced in this section should not also be applied to the slowest group of electrons in the runs at 0·52 mm. pressure, in spite of the fact that these were well checked in main features by three other sets of runs, data for which have not been published.

3. *The Boundary between the Negative Glow and Cathode Dark Space.*

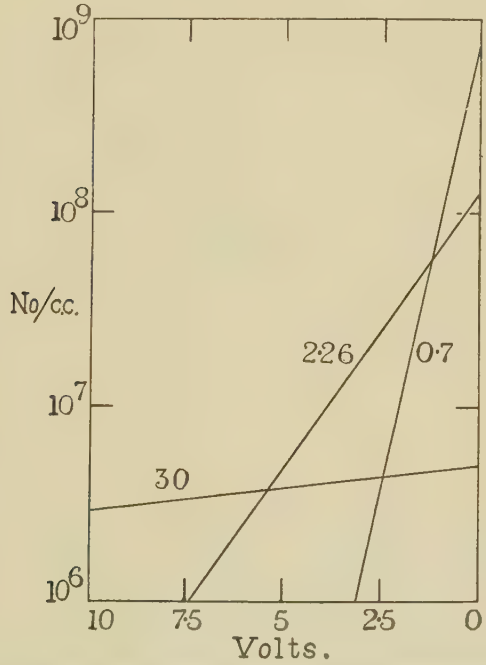
Although the collector characteristics for this region are not reliable as an indication of electron temperature and concentration, it is possible to estimate the latter by taking over the theory of Tonks and Langmuir for the boundary between a sheath and a plasma*. This can be done by assuming that the electron concentrations and temperatures are given correctly by the probe at the place of maximum concentration in the negative glow, interpolating a smooth potential curve between the negative glow and Faraday dark space curve found by the Langmuir method and the cathode dark space curve found by Brown and Thomson's or some other method, and applying Boltzmann's theorem individually to the electron groups found in the negative glow. This assumes that whatever agencies are responsible for productions of groups with an approximate Maxwellian distribution in the negative glow are able to maintain this distribution in the electric field retarding the movement of electrons into the cathode dark space. In the present cases there is no reliable direct information about the course of the potential curve in the cathode dark space, but the trend of the concentration curves can be shown by considering the electron concentrations, not as a function of position in the discharge, but as a function of the difference in potential between points in the boundary and the reference point in the negative glow.

Fig. 3 shows the results of such calculations for the discharge analysed in Table III. of the paper of Emeléus

* *Loc. cit.*

and Harris, at a pressure of 0·52 mm., based on the data for the point at a distance of 4·28 cm. from the anode, beyond which on the cathode side, there was, on the original interpretation of the characteristic curves, a large reversed field. To complete the data, it has been assumed that there was also a fast group of electrons present, with a temperature equivalent to 30 volts, and a concentration of 5.10^6 per c.c. at the reference point. The figure shows how

Fig. 3.



Graph to illustrate the different rates at which the concentrations of electrons in groups of different mean energies fall off in the retarding field in the boundary region between the negative glow and cathode dark space. The numbers against the lines give electron temperatures in volts.

rapidly the concentration of slow electrons falls off, whereas that of the fast electrons is little affected.

These results do not explain Seeliger's empirical rule that spectra which require greater energy for excitation appear relatively more prominently on passing from the negative glow into the cathode dark space *, since there is evidence

* Seeliger and Lindow, *Phys. Zeit.* xxvi. p. 393 (1925).

that the fast group alone is here responsible for the excitation of the radiation, mainly by non-cumulative processes, and the present treatment would leave its velocity distribution, on which the distribution of energy in the spectrum depends, unchanged. Failure to account for Seeliger's rule indicates rather that the theory is not valid when applied to the faster electrons, as is to be expected from the fact that they are probably responsible for transport of the major part of the tube current in this region.

Summary.

Analyses previously given for a cold cathode glow discharge in argon are reconsidered in the light of more recent knowledge of the properties of cold exploring electrodes. It is shown that although the information obtained in this way for the boundary region between the cathode dark space and the negative glow is not reliable, it is still possible to calculate the electron concentrations there by an application of Boltzmann's theorem. Some effects of secondary emission from the exploring electrode are discussed.

We have to thank Prof. E. V. Appleton, F.R.S., and Mr. E. W. Pike for their criticisms of this paper in manuscript.

XXXIX. *A Quantitative Thermo-electric Magnifier for Optical Pointers.* *By* J. H. Jeffree. (*From the Clarendon Laboratory, Oxford.**)

Abstract.

A simple device is described for magnifying deflexions of optical pointers, giving large magnification which is independent of fluctuations in illumination intensity.

Introduction.

A common defect in devices for the magnification of small deflexions of beams of light, whether by photoelectric or thermoelectric means, is that the magnification obtained depends on the intensity of illumination. This arises because the electrical effect produced is measured by reference to some external standard, such as a galvanometer. It could

* Communicated by Prof. F. A. Lindemann, Ph.D., F.R.S.

obviously be overcome by making the light itself provide the standard of reference, as for instance by the use of balanced photoelectric cells, but this might require elaborate apparatus, and another way is to use some device arranged to follow the movement of the beam itself, and then to observe the motion of the device. A method of doing this is described in this paper.

Description.

The instrument consists of a differential thermo-element connected to a loop of wire and suspended in a magnetic field, so that deflexions of the beam of light, which falls on the thermo-element, tend to cause rotation. The suspension fibre is very thin, so that its restoring force is negligibly small, and the thermo-element is mounted a short distance in front of the axis of suspension, so that it is displaced by rotation in the direction of the movement of the beam. The direction of the magnetic field is of course suitably chosen. In this way the rotation due to a small deflexion is enabled to continue until the thermo-element has regained its original relationship to the beam; that is to say, it follows the movements of the latter.

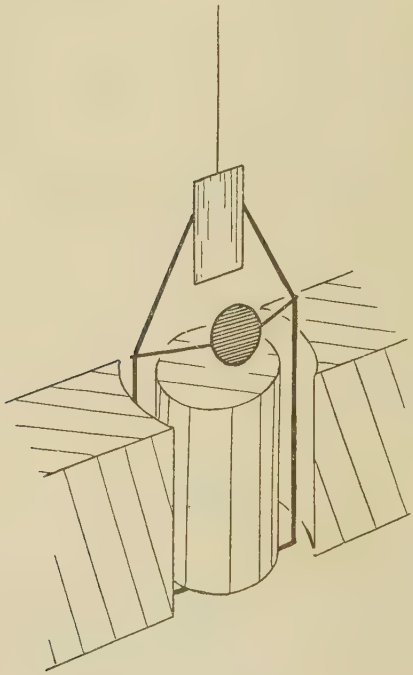
The movements of the thermo-element and loop could of course be observed in any convenient way, but in many cases the simplest method is to reflect back part of the beam from a small mirror fixed near the thermo-element, receiving it upon a scale in the usual way.

Construction.

The construction of the moving system is shown in the accompanying figure. The loop is of No. 40 bare copper wire, bent to form three sides of a rectangle 1·5 cm. high and 1 cm. wide. The thermo-element and connexions form the fourth side, at the top, and above this the ends of the wire project upwards to support the mirror, which is 3 mm. wide, 6 mm. high, and about 0·13 mm. thick. It is cemented to the wire projections with a little shellac. The quartz suspension fibre, 3 μ thick, is also similarly cemented to the upper end of the mirror.

A very simple form of differential thermo-element is used, consisting of a disc of constantan foil 8 mm. in diameter and 3 μ thick with copper wires, No. 46 S.W.G., soldered to it at the back on each side. These project radially from the disk to the loop, and extend 1 mm. on to the disc on either side, so that a gap of 1 mm. is left in the middle. Two opposed copper-constantan junctions are thus formed.

The thin copper wires are bent slightly forward, so that the disc is a small distance in front of the plane of the loop and of the axis of suspension. This distance determines the degree of magnification of the instrument, since the rotation will be greater, for a given displacement of the thermo-element, the nearer the latter is to the axis. It can conveniently be about 1 mm., giving a rotation of 1 radian for 1 mm. deflexion, which is of course doubled in the beam reflected from the mirror.



The loop swings between cylindrical pole pieces giving a field strength of about 1000 Gauss. These have a fairly wide clearance, and the suspension fibre is only 1 cm. long, so that no accurate levelling is required. The fastening of the upper end of the fibre is to a fine spring of No. 46 constantan wire, and the movement of the loop is limited by suitably placed stops so that no possible displacement can exert sufficient strain on the suspension to break it. The whole instrument thus becomes very robust and easy to use.

The above form is suitable for the magnification of galvanometer deflexions and similar purposes. It can, of course, be modified to suit any particular conditions of use.

Method of Use.

In use the magnifier is placed fairly near to the primary instrument whose deflexions are to be magnified—at a distance of 40 to 50 cm., for instance—and for convenience at a slightly higher level. The illumination is arranged so that a vertical image of a coiled filament, about 1 mm. wide, is projected from the primary instrument on to the thermo-element and mirror of the magnifier and the reflected beam then passes back over the primary instrument, and is received on a scale behind it. This is usually the most convenient arrangement, as the slightly upward path of the beam ensures that when, after reflexion from the mirror of the magnifier, it passes out through the window of the latter, no reflexions are thrown back from this window on to the thermo-element. Such reflexions might cause uneven magnification. Alternatively, the window may be suitably inclined to prevent such reflexions.

The window may take the form of a lens, of suitable focal length to produce on the scale an image of the mirror of the primary instrument, or of some indicator placed in the path of the primary beam, such as a vertical wire.

Magnification can be varied by changes in the working distance. The instrument has critical magnetic damping for a certain intensity of illumination, and for intensities above this level the period of swing is shortened without affecting the damping, so that the time taken to reach a final reading is not changed. For the form described this time is about two seconds. For weaker illumination the response is slower, the instrument being over-damped, and eventually the magnification falls off as the controlling force of the fibre becomes of importance.

An interesting phenomenon occurs with very intense illumination. The period of oscillation becomes quite short and comparatively undamped, and the heat capacity of the junctions is then sufficient to delay the response appreciably. A vigorous oscillation is set up, each junction as it is carried into the beam being strongly heated, and therefore reversing its direction of swing. This does not occur, however, till intensities are reached much higher than are needed for satisfactory working, and therefore causes no trouble in practice.

Theoretical.

One or two points may be mentioned in connexion with the theoretical design of the instrument.

It is unlikely that any material advantage would result from the use of a more elaborate thermo-element. Nothing is to be gained by increasing the number of junctions, since the resistance of the moving loop can have any desired value to match that of the thermo-element. The available electrical energy would not be increased thereby. Nor would any great difference be made if the two junctions had separate receiving surfaces instead of the single disk of constantan, and were connected by a wire, since the extent of the surfaces would have to be the same, being fixed by considerations of convenience. The thermal conductivity between the junctions varies with the electrical conductivity for a given choice of materials, and the best compromise between the two is attained as easily with the simpler arrangement as with a more complicated one.

The accurate calculation of the efficiency of the thermo-element is rather complicated, and need not be discussed. An estimate of the order of the effect, which may be made on the lines indicated below, is generally sufficient; for any errors introduced thereby have the same effect as changes in the light intensity.

If light of intensity I ergs per second falls centrally between the junctions in the form of a sharp line, a temperature gradient is set up whose value is $\frac{I}{2C}$, where C is the heat conductivity of unit length of the surface. This gradient decreases from the centre outwards as heat is lost through radiation and convection, but if the junctions lie within the range where it is sensibly constant a very small deflexion D may be considered as simply displacing the centre of the temperature gradient, giving a differential heating of the junctions of $2D \times \frac{I}{2C} = \frac{ID}{C}$. Actually, the gradient falls off from the centre according to a hyperbolic cosine function of $L\sqrt{\frac{R}{C}}$, where L is the length, R the rate of loss by radiation and convection per unit length of surface per degree rise of temperature, and C the heat conductivity as above; but when the surface extends beyond the junctions to a distance as great as the separation between them, the calculation may be made without introducing serious error on the assumption that the strip extends indefinitely on either

side of the centre. A logarithmic law then holds, the decrement of the gradient for unit length L being equal to $e^{-L\sqrt{\frac{R}{C}}}$. This of course neglects the effect of the connecting wires, but it serves to indicate a lower limit to the thickness of the receiving surface, beyond which the temperature gradient at the junctions would be considerably less than that at the centre, where it is inversely proportional to the thickness. Such considerations show that if the separation of the junctions is to be 1 mm., a thickness of 3μ of constantan foil is suitable for the receiving surface, the temperature gradient at the junctions then having a value of about half that at the centre. A much greater thickness would begin to affect the time of response through its heat capacity.

The electrical part of the design follows that of moving coil galvanometers, except that the torsional restoring force is not immediately known. It is equal, for a deflexion of one radian, to the electromagnetic torsion theoretically produced by a deflexion of the beam equal to the distance to which the thermo-element is displaced in front of the axis of rotation. It is therefore, proportional to this distance, for a given light-intensity, and therefore inversely proportional to the magnification of the instrument. The light-intensity required will, in fact, be proportional to the magnification. It is, however, advantageous to use a high magnification, and to shorten the working distance of the magnifier from the primary instrument, as the available intensity of illumination increases in practice inversely as the square of the working distance.

The thickness of the suspension must be such that its restoring force is small in comparison with the electromagnetic forces. In the instrument as described above it provides less than 4 per cent. of the restoring force, when the light intensity is sufficient to give a critically damped response of about 2 seconds period. The necessary intensity of the image is then about 5000 ergs per centimetre length.

Conclusion.

The instrument described above has been used during the past year, in various forms, for the magnification of galvanometer readings and other small deflexions, with very satisfactory results. It is easy to use, and the magnification is constant at different times, approximately linear over a large range and calculable from the constants of the instrument. The construction is also comparatively easy. The magnification

is sufficient to reach the "Brownian motion" limit with fairly sensitive galvanometers, without the use of special arrangements for illumination. It is also very little affected by stray radiation or other external influences, and can therefore be used in an ordinary well lighted room. The design is capable of modification over a wide range to meet special circumstances.

The author would like to express his indebtedness to Professor F. A. Lindemann for the assistance and laboratory facilities which enabled this device to be worked out.

XL. *On "Matthiessen's Constant," or the relation between Electrical Resistivity and Temperature Coefficient of Metals.*

By J. T. MacGregor-Morris, *M.I.E.E.*, *and* R. P. Hunt, *B.Sc.*(*Eng.*) *.

Contents.

i. Introduction.
ii. Gradual Appreciation of General Applicability.
iii. Exceptions.
iv. Physical Meaning of the Law.
v. Conclusion.
vi. Sources of Information on the Subject.

i. *Introduction.*

THE fact that the electrical resistivity of a metal and its temperature co-efficient are related was first noticed by Dr. Matthiessen in 1862, in the course of his search for a material suitable for standard resistances. In the British Association Report on Electrical Standards for that year [1] he stated that as the result of tests on a number of pure metals and alloys he had observed the following law to be maintained in most cases:—

"The percentage decrement between 0° and 100° C. in the conducting power of an alloy in a solid state stands in the same ratio to the mean percentage decrement of the components between 0° and 100° C as the conducting power of the alloy at 100° C. does to the mean conducting power of the components at 100° C.

"In other words, the absolute difference in the observed resistance between 0° and 100° C. of an alloy is equal to the

* Communicated by the Authors.

absolute difference between the means of the resistance of the component metals between 0° and 100° C."

The first of the above statements may be written :

$$\frac{P}{P'} = \frac{C}{C'},$$

where P and P′ are the actual and calculated percentage decrements respectively, between 0° and 100° C., and C and C′ are the actual and calculated conductivities respectively, at 100° C.

The second statement may be written :

$$R_{100} - R_0 = R'_{100} - R'_0,$$

where R_{100} and R_0 are the actual resistances at 100° and 0° C. respectively, and R'_{100} and R'_0 are the calculated mean resistances of the components at 100° and 0° C. respectively.

It can easily be shown that the above two statements are equivalent.

It is now known that the rule breaks down in the case of many alloys, in fact, there are some, such as constantan and manganin, which show practically no change in resistivity over the range of temperature mentioned, although the resistivity of the component metals separately may increase by 40 or 50 per cent. over the same range. For impure metals which contain only a small amount of alloying material, however, the law holds generally, and the authors would restate it in the following form :—

"For the alloys of any particular metal with another the rate of increase of resistivity with temperature is independent of the concentration.

"That is, $\frac{dR}{dT}$ is the same for the various alloys."

If $$R = R_0(1 + \alpha t + \beta t^2),$$

then $$\frac{dR}{dT} = R_0(\alpha + 2\beta t),$$

and is independent of concentration, where

R = resistivity at $t°$,

R_0 = resistivity at 0° C.,

α and β = constants as regards the temperature.

$R_0 \times$ temperature coefficient is also a constant for the given alloys. We shall refer to the law in this form as Matthiessen's Law.

The authors suggest that it is appropriate to attach the name of Matthiessen to this constant and that it should be known as the "Matthiessen Constant" for each metal. It would thus perpetuate the name of its discoverer and honour the man who did so much pioneer work on the conductivity of metals and alloys. The law as stated above is found to hold over a much wider range of temperature than that considered by Matthiessen, and also applies when other physical changes besides alloying are produced in the metal.

ii. *Gradual Appreciation of General Applicability.*

Matthiessen's work in this direction appears to have been forgotten for many years, and the present authors cannot find that it is referred to again until Dr. J. A. Fleming (now Sir Ambrose Fleming), mentioned it in a lecture before the Royal Institution in 1896 on the results of his experiments with Professor Dewar on the resistivity of metals and alloys at normal and low temperatures. He stated: "If to one pure metal we add a small quantity of any other metal, the result is always to raise the resistance line almost parallel to that of the predominant constituent. This is equivalent to saying that the effect of the added material is to increase the specific resistance, but not to alter the slope or form of the curve."

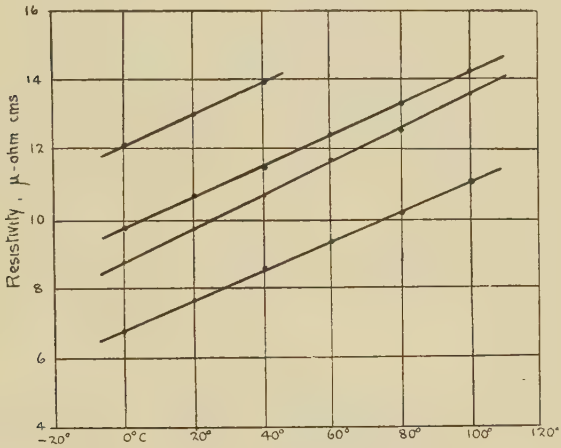
Some of Fleming's work[(2)] on various specimens of iron and nickel is shown in figs. 1 and 2, together with examples from other observers for comparison. It is seen that the lines are nearly parallel to each other, although the individual resistivities are widely different. Fleming also found the rule to hold when the change in the metal is simply due to alteration in the crystalline structure, as in the case of hard-drawn and annealed silver and copper *.

Professor John Hopkinson, in the course of his work on the magnetic properties of iron alloys, about 1890, examined a steel containing 24·5 per cent. of nickel[(3)], which exhibited some extraordinary properties. It was found to exist in two or more different states: normally, it is only paramagnetic, but after cooling to $-51°$ C. it becomes ferromagnetic, and also has a much lower resistivity, the change being from 72 to 52 microhm cm. at 0° C. The resistivity/

* Dr. D. K. Morris and one of the present authors carried out the major portion of the experimental work in Fleming and Dewar's researches on pure metals and alloys, and the parallelism of the resistivity lines impressed itself upon these experimenters. In 1899, the latter prepared a list of the values of this constant for many of the elements, and has dealt with it each year in his lectures at East London College.

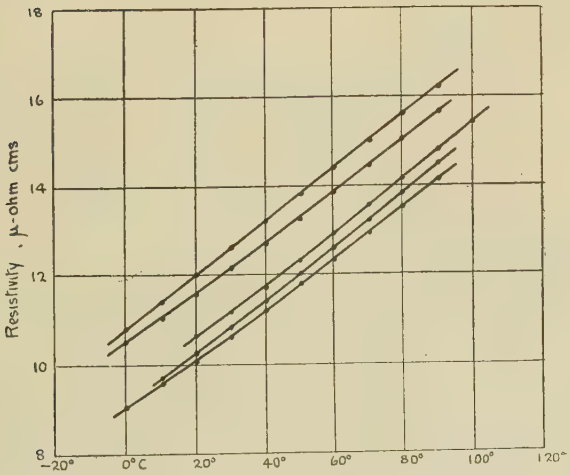
temperature curves, shown in fig. 3, are nearly parallel, however, over the range of temperature 0° to 400°C., showing that the law still applies, even though there is a

Fig. 1.



Nickel, various samples (Fleming, Lees).

Fig. 2.



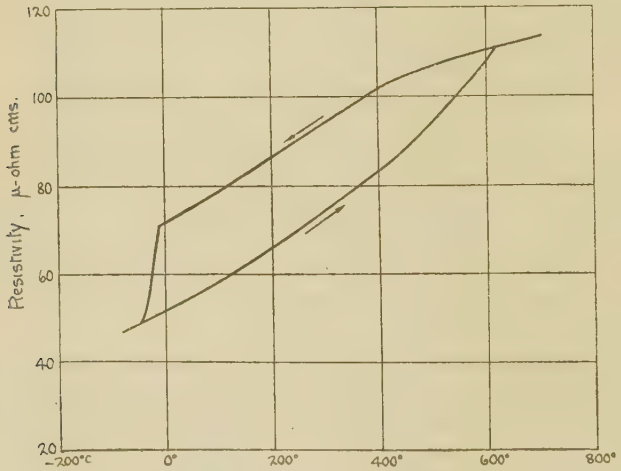
Iron, various samples (Fleming, Nicolai).

radical change in the molecular state of the metal. Other nickel steels were found to exhibit the same properties, and also a manganese steel.

The application of Matthiessen's law was later confirmed by other observers for particular metals.

In 1911, J. H. Dellinger[4] carried out a series of tests on a large number of specimens of high conductivity copper and aluminium from various sources, which were published by the American Bureau of Standards, with a view to the establishment of international standard values for the resistivity and temperature coefficient for these metals. The work showed conclusively that the temperature coefficient varied inversely as the resistivity, and that as a result the increase in the latter per degree is a constant, whatever the purity of the metal.

Fig. 3.



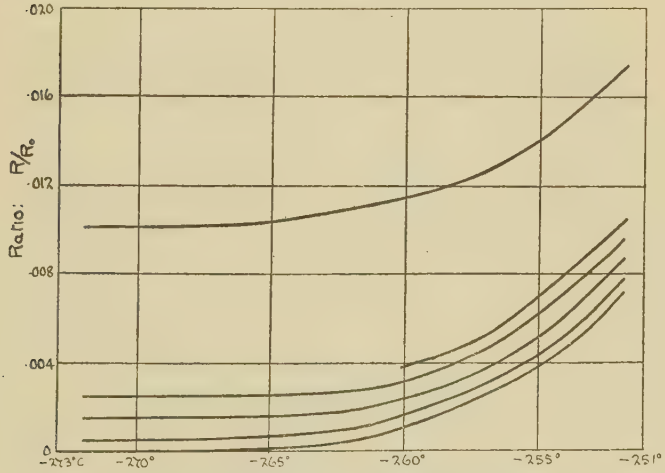
Nickel-steel (Hopkinson).

Dellinger points out the practical importance of the law, and states that it may be used to determine the resistivity of an irregularly shaped rod by measurement of the temperature coefficient alone. The resistivity is obtained by dividing the constant for the metal by the observed temperature coefficient; the latter can be measured without knowing the dimensions of the rod, and hence the method has the double advantage of greater certainty and economy of time.

Kammerlingh Onnes[6], in his work at extremely low temperatures, found the rule to apply over the whole range of temperature investigated, from near absolute zero to normal temperatures. The only exception he found was that when certain of the metals were superconducting, their

resistance/temperature curves were coincident. He suggested that the resistance of a metal might be composed of two parts, one an "ideal" resistance, characteristic of the pure metal itself, which is a function of temperature and tends to become zero at the absolute zero of heat, and the other, an "added" resistance due to impurities or to strain, practically independent of temperature over a considerable range, but dependent on the nature and amount of impurity and degree of strain. The curves for a number of specimens of gold are shown in fig. 4 ; it is seen that they are similar in shape and differ only in that they are displaced various distances vertically above the horizontal axis.

Fig. 4.



Gold, various specimens (Onnes, Meissner, etc.).

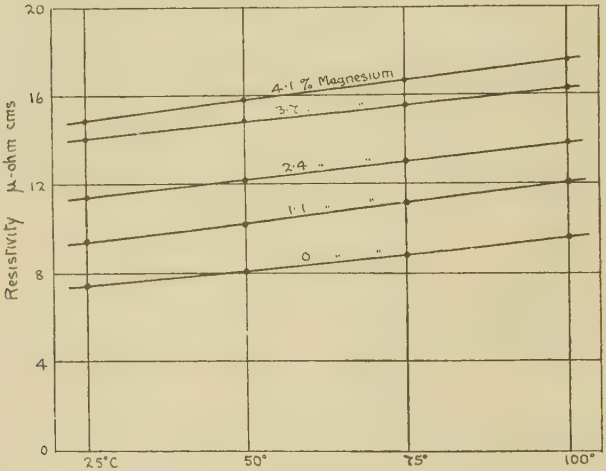
Other observers including Clarke, Forde, and Taylor in 1899, Feussner and St. Lindeck [5], Urazov in 1912 (see fig. 5), and Grassi in 1918, have made contributions to the subject.

The work of Hanson and Marryat [7] at the National Physical Laboratory should be mentioned; also that of Guertler and Schulze [8] in Germany. Two examples of the work of the latter are shown in fig. 6. Further, they showed that when iron is alloyed with gold the limit is about 2 per cent. The numerical values of the quantities concerned in the latter example are shown in Table I.

An important contribution was made by Geiss and van Liempt [9] at the Philips lamp works at Eindhoven, Holland, in 1925. They announced definitely that Matthiessen's law

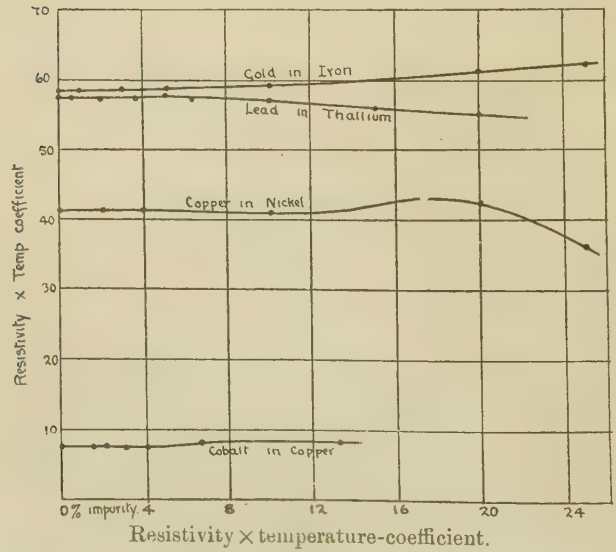
always holds for small percentages of impurity, and in the case of the tungsten-molybdenum alloys for the complete

Fig. 5.



Cadmium alloyed with magnesium (Urazov).

Fig. 6.



Resistivity × temperature-coefficient.

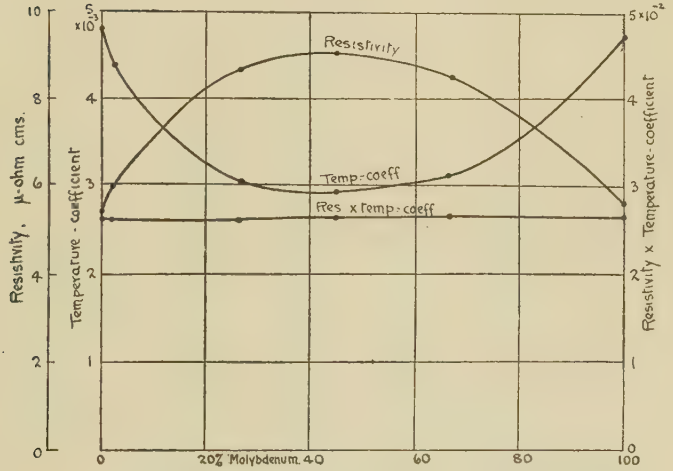
range of compositions from 100 per cent. of one to 100 per cent. of the other, as shown in fig. 7 and Table II.

TABLE I.

Iron alloyed with gold.

Per cent. Gold.	Resistivity at 0° C.	Temp. coeff. at 0° C.	Res. × Temp. coeff.
0	11·0	$5{\cdot}3\times10^{-3}$	$58{\cdot}3\times10^{-3}$
1	11·9	4·9 „	58·3 „
3	14·3	4·1 „	58·6 „
5	16·3	3·6 „	58·7 „
10	19·7	3·0 „	59·1 „
20	17·0	3·6 „	61·2 „
40	20·0	3·3 „	66·0 „

Fig. 7.



Tungsten–molybdenum alloys (Geiss and van Liempt).

TABLE II.

Tungsten and molybdenum alloys.

Atomic per cent.		Resistivity at 18° C.	Temperature coefficient.	Res. × Temp. coefficient.
Tungsten.	Molybdenum.			
100·0	·0	5·45	$4{\cdot}82\times10^{-3}$	$26{\cdot}2\times10^{-3}$
98·1	1·9	5·97	4·37 „	26·1 „
73·2	26·8	8·65	3·04 „	26·3 „
55·8	44·2	9·06	2·91 „	26·4 „
33·5	66.5	8·52	3·12 „	26·6 „
·0	100·0	5·58	4·73 „	26·4 „

In 1928, McLennan[10], of the University of Toronto, obtained specimens of two metals which are only produced in the pure state with great difficulty, namely, cobalt and chromium. He found that annealing these metals for several hours had the effect of shifting the resistivity/temperature curves downwards parallel to themselves, thus showing another illustration of Matthiessen's law. McLennan suggested that the decrease in resistance was due to occluded gas being driven off by the annealing. A similar decrease was observed in lead after it had been cooled down to a temperature of 7° absolute. The law was found to hold for lead alloyed with cadmium up to 1 per cent. of added metal, over the range of 200° to 300° absolute.

A metal may exist in either the monocrystalline or the polycrystalline forms, the latter being the usual condition. It has been shown by Kannuluik[11] and Laby and others, that in the case of copper, the constant is the same whether the metal is in the form of one crystal or many, over a considerable range of temperature. There is also evidence that the law holds good both when the impurity is another metal and when it is a non-metal, such as phosphorus or sulphur. Gaseous impurities can probably be included also, as mentioned above.

The limits of temperature over which the law is applicable are of interest; in the case of nickel, departures from the law begin to occur above 250° C.; this limit probably varies in different metals. The limiting percentage of alloying material also varies widely in different pairs of metals, and probably depends on the limits of solution of one metal in the other. The relative positions of the components in the Periodic Table may also be a controlling factor, but further investigation is required to settle these points.

In Table III. the values of Matthiessen's constant for a number of metals are given for the temperatures 0° and 100° C.

iii. *Exceptions.*

While there is ample evidence of a very wide application of the law, there are a few cases of changes of physical state which may be produced in a metal where the law is not obeyed.

The work of several experimenters on the resistance of sputtered metal films of extremely small thickness shows that the law definitely does not hold in these cases. The change of resistivity with temperature may be of any

amount, and either positive or negative ; it appears that the metal is in a totally different form from that of the bulk metal.

A particular class of alloys is that of amalgams of mercury with other metals. These have been extensively studied by

TABLE III.

From International Critical Tables.				Matthiessen's Constant.	
Metal.	R_0.	α_0.	β.	At 0° C. $R_0\alpha_0$.	At 100° C. $R_0(\alpha+200\beta)$.
Aluminium ...	2·62	4·46	1·8	11·67	12·63
Antimony	39·0	4·14	6·2	161·5	210·0
Bismuth	106·5	3·91	5·8	416	540
Calcium	4·27	3·64	—	15·53	15·53
Cadmium	6·83	3·73	3·7	25·46	30·50
Cæsium	18·83	4·78	—	90·0	90·0
Cobalt	5·59	5·51	5·3	30·8	36·7
Copper	1·55	4·35	0·4	6·76	6·88
Gallium	53·4	3·46	—	184	184
Gold	2·19	3·65	—	8·00	8·00
Indium	8·37	4·77	—	39·9	39·9
Iridium	6·08	3·67	—	22·3	22 3
Iron	8·53	7·26	9·6	61·9	78·4
Lead	19·8	3·95	2 6	78·2	88·7
Lithium	8·55	4·75	1·0	40·6	42·4
Magnesium ...	4·27	3·88	0·7	16·6	17·2
Mercury	94·1	·90	0·8	84·6*	100·7*
Molybdenum...	5·14	4·79	0·3	24·6	25·0
Nickel	6·93	5·44	6·0	37·7	46·0
Palladium	10·0	3·61	—	36·1	36·1
Platinum	9·83	3·98	−0·5	39·1	37·9
Potassium	6·15	5·81	15·3	35·7	64·0*
Rhodium	4·70	4·23	−1·8	19·9	18·1
Rubidium	11·3	5·53	—	62·6	62·6
Silver	1·50	3·89	—	5·85	5·85
Sodium	4·2	5·09	6·1	21·3	32·0*
Tantalum	13·8	3·3	—	45·5	45·5
Thallium	17·65	4·05	—	71·5	71·5
Tin	10·48	4·36	2·4	45·7	50·8
Tungsten	5·00	5·24	0·7	26·2	26·9
Zinc	5·64	3·47	1·2	19·5	20·9
Zirconium	164	1·16	−0·6	190	169

In those cases where no figures are given the value of β is small.

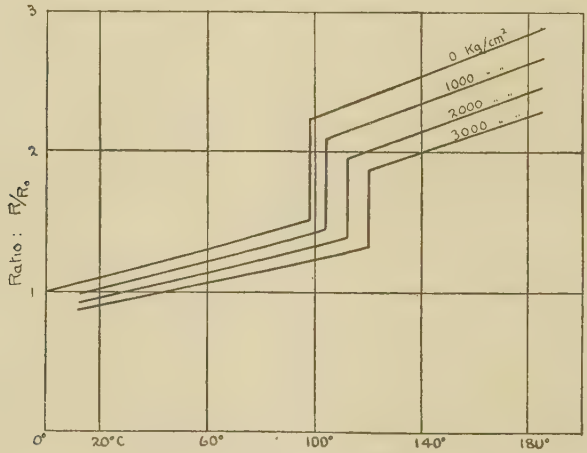
* In the liquid state.

Prof. Evans and his colleagues[12] at University College, Swansea. The results indicate that the law is not obeyed by these alloys, at any rate, for appreciable percentages of added metal. It is suggested that in many cases inter-

metallic compounds are formed, which may account for the deviations from the law.

The experiments of Bridgman [13] at Harvard on the effect of pressure on the resistivity of metals indicate that, in a few cases, such as sodium and strontium, the law is followed up to moderate pressures, such as 1000 Kg/cm.^{2} (6 tons/in.^{2}), but that for the majority of metals, there is deviation from the law as soon as pressure of the amount mentioned is applied. It is possible that the law holds even in these cases for pressures of smaller intensity. The resistivity/temperature curves for sodium under different pressures are shown in fig. 8. It will be seen that the law is followed

Fig. 8.



Sodium under pressure (Bridgman).

when the metal is in both the solid and liquid states at moderate pressures.

The change in resistance of a metal when in a magnetic field has been studied extensively; the work of Fleming and Dewar, Kapitza [14], and others, have shown that Matthiessen's law is not followed under these conditions. The increase in resistance of the metal is found to be very much greater at low temperatures than at high, so that the slope of the resistivity characteristic is completely altered.

iv. *The Physical Meaning of the Law.*

It is of interest to inquire into the physical significance of the law and whether it can be proved on theoretical grounds.

Kammerlingh Onnes's theory of two component resistances in every metal, an "ideal" and an "added" resistance covers the observed facts, but a comprehensive theory of electrical conduction has not yet been established. It should be mentioned, however, that W. V. Houston [15] has recently claimed that a qualitative explanation of the law can be given on the basis of the Sommerfeld theory of Wave Mechanics.

The whole subject is, at present, in a very controversial position.

v. *Conclusion.*

From the foregoing it will appear that this constant has a decided practical value in that by its use one is able to predict rapidly the resistivity of a sample of commercially pure metal by means of experiments determining its resistance temperature coefficient; no knowledge of the dimensions of the specimen being necessary.

It is true that though there are limitations to the extent of the application of "Matthiessen's constant," such as the limits set by the formation of eutectic mixtures, yet from the examples quoted it is clear that the law is of wide applicability, and that each metallic element has associated with it a definite constant, which remains unaltered, though the metal may be subjected to a wide range of physical conditions, and also may have chemical impurities added to it.

vi. *Sources of Information on the Subject.*

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(7) Hanson and Marryat, Jour. Inst. Met., xxxvii. (1927).
(8) Guertler and Schulze, International Critical Tables, vi. p. 124.
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(11) Kannuluik, Proc. Roy. Soc. cxxxi. p. 320 (1931).
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(13) Bridgman, 'The Physics of High Pressure' (1931).
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March 1932.

XLI. *The Stark Effect for Strong Fields.* *By* M. A. El-Sherbini, *B.Sc., The Faculty of Science, The Egyptian University, Cairo*.*

§ 1. *Introductory.*

In a previous communication † third-order terms in the theory of the Stark effect were obtained. Since the communication of that paper my attention has been drawn to a paper by Y. Ishida and S. Hiyama ‡, in which the results of experiments with fields in the neighbourhood of 7×10^5 volt/cm. are recorded. Their results give measurable effects arising from third-order terms. In view of this I have revised the work given in my last paper §, and find that, owing to a slip in the calculation, equations (20) and (21) should read

$$\Delta_3 E = -\frac{4k^2}{\mu}(\alpha_0\alpha_3 + \alpha_1\alpha_2)D^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$\Delta_3 E = -\frac{3D^3\hbar^{10}}{2^{15}\mu^5 Z^7 e^{11}\pi^{10}}(m-n)(m+n+s)^7$$
$$\{23(m+n+s)^2 - (m-n)^2 + 11S^2 - 22S + 50\}. \quad (21)$$

respectively.

This result now agrees with that obtained by S. Doi and quoted in Ishida and Hiyama's paper, but which I have unfortunately, not been able to see in the original.

As a further check on my calculations I have solved the same problem by a different method, and find that the same result is obtained. This method was originally indicated by G. Wentzel ‖, and was used by him in obtaining first- and second-order terms.

In the following I give the main steps in the calculation of the third-order terms by this method.

§ 2. *Third-Order terms.*

The two wave equations (5) given in my last paper can be reduced to the form

$$\psi'' + \frac{1}{x}\psi' + \frac{4\pi^2}{\hbar^2}\left(a + \frac{b}{x} + \frac{c}{x^2} + dx\right)\psi = 0, \quad . \quad (1)$$

* Communicated by Prof. A. M. Mosharrafa D.Sc.

† Phil. Mag. xiii. p. 24 (Jan. 1932).

‡ Sc. Papers of Ins. Phys. Ch. Research, Tokyo, no. 151 (July 20, 1928).

§ *Ibid.*

‖ G. Wentzel, *Zeit. f. Phys.* xxxviii. p. 518 (1927).

where

$$a=2\mu\mathrm{E}, \quad b=\mu(Ze^2\pm\beta), \quad c=\frac{1}{4}\left(\frac{lh}{2\pi i}\right)^2, \text{ and } d=\pm 2\mu e\mathrm{D}; \quad . \quad . \quad . \quad (2)$$

μ, e are the mass and charge of the electron, D is the strength of the field, l a quantum number takes any value 1, 2, 3 ..., and β is a constant of separation.

Applying the substitution

$$\psi=e^{\frac{2\pi i}{h}\int y\,dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

equation (1) can be transformed to the Riccati equation

$$\frac{h}{2\pi i}\left[y'+\frac{y}{x}\right]=\left[a+\frac{b}{x}+\frac{c}{x^2}+dx\right]-y^2, \quad . \quad (4)$$

whose solution* is

$$y=\sum_{\nu=0}^{\infty}\left(\frac{h}{2\pi i}\right)^{\nu}y_\nu. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Then we have as a condition† the relation

$$\sum_{\nu=0}^{\infty}\left(\frac{h}{2\pi i}\right)^{\nu}\oint y_\nu\,dx=kh. \quad . \quad . \quad . \quad . \quad (6)$$

Obtaining the successive values of y up to y_3 and substituting in (6), we get

$$\begin{aligned}kh=2\pi i\Bigg\{&\left[\frac{1}{2}ba^{-1/2}-\frac{1}{2}\frac{h}{2\pi i}-\sqrt{c}\right]\\&+d\left[\frac{3}{16}b^2a^{-5/2}-\frac{1}{4}ca^{-3/2}+\frac{1}{16}\left(\frac{h}{2\pi i}\right)^2a^{-3/2}\right]\\&+d^2\left[\frac{35}{128}b^3a^{-9/2}-\frac{15}{32}bca^{-7/2}+\frac{25}{128}\left(\frac{h}{2\pi i}\right)^2ba^{-7/2}\right]\\&+d^3\Bigg[\frac{33\times35}{16\times128}b^4a^{-13/2}-\frac{9\times35}{2\times128}cb^2a^{-11/2}+\frac{35}{128}c^2a^{-9/2}\\&\quad+\frac{35\times21}{8\times128}\left(\frac{h}{2\pi i}\right)^2b^2a^{-11/2}-\frac{35\times3}{2\times128}\left(\frac{h}{2\pi i}\right)^2ca^{-9/2}\\&\quad+\frac{35\times5}{16\times128}\left(\frac{h}{2\pi i}\right)^4a^{-9/2}\Bigg]\Bigg\}. \quad . \quad . \quad . \quad . \quad . \quad (7)\end{aligned}$$

Put $k=m$ when $b=\mu(Ze^2+\beta)$ and $d=2\mu e\mathrm{D}$,
and $k=n$ when $b=\mu(Ze^2-\beta)$ and $d=-2\mu e\mathrm{D}$. } . (8)

* See Wentzel's paper, quoted above.
† *Ibid.* p. 521.

Subtract and add the two resulting equations, and then make the substitution

$$\left.\begin{aligned} \mathrm{E} &= \mathrm{E}_0 + \Delta_1\mathrm{E} + \Delta_2\mathrm{E} + \Delta_2\mathrm{E}, \\ \beta &= \beta_0 + \mathrm{D}\beta_1 + \mathrm{D}^2\beta_2 + \mathrm{D}^3\beta_3, \\ s &= l + 1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

Then, picking out the various coefficients of D^0, D, and D^2 from the former, we get

$$\left.\begin{aligned} \beta_0 &= \frac{Ze^2(m-n)}{(m+n+s)}, \\ \beta_1 &= -\frac{h^4(m+n+s)^2}{2^6\mu^2Z^2e^3\pi^4}\mathrm{N}, \\ \beta_2 &= \frac{h^8(m-n)(m+n+s)^5}{2^{11}\mu^4Z^5e^8\pi^8}\mathrm{N}', \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

where

$$\left.\begin{aligned} \mathrm{N} &= 3(m+n+s)^2 - 3(m-n)^2 - (s-1)^2 + 1, \\ \mathrm{N}' &= (m+n+s)^2 - (m-n)^2 - 6(s-1)^2 + 6. \end{aligned}\right\} . \quad (11)$$

Then, picking out the coefficients of D^3 from the latter, after using the relations (10) and (11) and the values of $\Delta_1\mathrm{E}$ and $\Delta_2\mathrm{E}$, we get for third-order terms

$$\Delta_3\mathrm{E} = -\frac{3h^{10}(m-n)(m+n+s)^7\mathrm{D}^3}{2^{15}\pi^{10}\mu^5e^{11}\mathrm{Z}^7}$$
$$\{23(m+n+s)^2 - (m-n)^2 + 11s^2 - 22s + 50\}. \quad (12)$$

My thanks are due to Prof. A. M. Mosharrafa, under whose direction the above work was done.

XLII. *A Newtonian Gravitational System and the Expanding Universe.* *By* W. R. Mason *.

Summary.

The general relativity theory, when applied to the cosmology of the universe, yields a picture which is not in agreement with experiment, since it requires a time duration far less than that required for the evolution of the stars. The present paper shows that a Newtonian gravitational system, consisting

* Communicated by the Author.

of an all pervading gas obeying Boyle's law and having a Newtonian inverse square law attraction between its individual parts, is capable of giving a consistent explanation of the red shift of the light from the distant galaxies.

It is shown that there are two stable static density-space solutions for the medium, both of which require an infinite amount of matter for the whole universe. As a consequence this theory—in agreement with the general relativity theory—shows that a finite distribution of matter is unstable and will be propagated away from the centre of the universe, at first slowly and then with an increasing velocity which finally approaches the velocity of light.

Assuming an infinite amount of matter for the whole universe, distributed initially in any manner except that given by the static solutions, it is shown that the density of the medium will have a periodic space-time solution whose period is $T = \sqrt{\pi/k\rho_0}$, k being the gravitational constant and ρ_0 the average density of matter at the centre of the universe. The velocity of recession of one part of the medium from another is calculated, and is shown to obey Hubble's law for any distance so far observed. The average numerical value of ρ_0 is found to be approximately 10^{-26}. This gives for the period T a value of 2×10^9 years. Comparing the density ρ_0 with Hubble's value of the average density of matter in the galaxies, it is found that the value of the ratio of distributed matter to concentrated matter is in the order of 10^5. Finally, assuming that the stars are carried along on the average in this periodic movement of the medium, an explanation of the red shift is obtained which allows sufficient time for the evolution of the stars.

1. *Derivation of Equations of Motion.*

THE Newtonian mechanical system considered here is a very simple one. It consists of the universe filled with a perfect gas obeying Boyle's law, the individual parts of which have a Newtonian inverse square law attraction acting between them. The equations of motion of the system are obtainable from a Lagrangian equation and a continuity equation. Suppose that we consider an element of volume $dx\,dy\,dz$ of some coordinate system. Then with respect to this volume the kinetic energy of the gas will be

$$T = \tfrac{1}{2}\rho(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,dx\,dy\,dz, \quad . \quad . \quad . \quad . \quad (1)$$

where T is the kinetic energy, ρ the density of the medium,

and $\dot{x}$, $\dot{y}$, $\dot{z}$ the velocities of the gas in the x, y, z directions respectively. By Lagrange's equation

$$\left[\frac{\partial}{\partial t}\left[\frac{\partial \mathrm{T}}{\partial \dot{x}}\right]-\frac{\partial \mathrm{T}}{\partial x}\right] dx\,dy\,dz=\mathrm{F}_x=\text{applied force in the } x \text{ direction. . . (2)}$$

Since ρ is not a function of the velocity in Newtonian mechanics, we have

$$\frac{\partial}{\partial t}(\rho\dot{x})\,dx\,dy\,dz=\mathrm{F}_x. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The force in the x direction may be calculated as follows. Since the gas obeys Boyle's law, we can write

$$pv=c^2m,$$

where p is the pressure, v the volume under consideration, m the mass of material in the volume, and c^2 a constant. We have also

$$\rho v=m.$$

Dividing, there results

$$\frac{p}{\rho}=\frac{c^2m}{m}=c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The applied force in the x direction of the element of volume will be

$$(p_1-p_2)\,dy\,dz=\mathrm{F}_x,$$

where p_1 is the pressure of the medium when $x=x_0$ and p_2 when $x=x_0+dx$. Hence

$$\mathrm{F}_x=-\frac{\partial p}{\partial x}dx\,dy\,dz=-c^2\frac{\partial \rho}{\partial x}dx\,dy\,dz. \quad . \quad . \quad (5)$$

Similarly if V is the Newtonian gravitational potential due to the distribution of matter throughout the universe, the force due to the gravitational attraction is

$$-\rho\frac{\partial \mathrm{V}}{\partial x}\,dx\,dy\,dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence (3) becomes

$$\frac{\partial}{\partial t}(\rho\dot{x})=-\left[c^2\frac{\partial \rho}{\partial x}+\rho\frac{\partial \mathrm{V}}{\partial x}\right], \quad . \quad . \quad . \quad . \quad (7)$$

with two similar equations for the y and z direction. These equations can be combined into one vector equation,

$$\frac{\partial}{\partial t}(\rho\dot{\mathbf{r}})=-[c^2\nabla\rho+\rho\nabla \mathrm{V}]. \quad . \quad . \quad . \quad . \quad (8)$$

The equation of continuity states that matter cannot be created or destroyed at any point in the medium. Embodied in mathematical form, this is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \dot{\mathbf{r}}) = 0. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

These two equations determine the motion of matter at any point. By eliminating the momentum $(\rho \dot{\mathbf{r}})$ there results one equation,

$$\frac{\partial^2 \rho}{\partial t^2} = c^2 \nabla^2 \rho + \operatorname{div}(\rho \nabla \mathrm{V}). \quad . \quad . \quad . \quad . \quad (10)$$

2. *The Relativity Principle.*

In the absence of any potential gradient, $\nabla \mathrm{V}$, equation (10) shows that the medium is capable of propagating a disturbance with a velocity c. The question at once arises whether this type of a system will be governed by the same law when a transfer is made to another coordinate system which is travelling with a constant velocity u with respect to the first system. Since this is a purely Newtonian system, this transference can be made.

If x', y', z' are coordinates of the new system, then

$$x' = x - ut, \; y' = y \; ; \; z' = z \; ; \; \frac{d}{dx'} = \frac{d}{dx} \; ; \; \frac{d}{dy'} = \frac{d}{dy} \; ; \; \frac{d}{dz'} = \frac{d}{dz}.$$

The momentum of the material in the x direction inside the element of volume $dx\,dy\,dz$ will be

$$\rho(\dot{x}' + u).$$

Hence

$$\left.\begin{aligned} \frac{\partial}{\partial t}[\rho(\dot{x}' + u)] &= -\left[c^2 \frac{\partial \rho}{\partial x'} + \rho \frac{\partial \mathrm{V}}{\partial x'}\right], \\ \frac{\partial}{\partial t}[\rho \dot{y}'] &= -\left[c^2 \frac{\partial \rho}{\partial y'} + \rho \frac{\partial \mathrm{V}}{\partial y'}\right], \\ \frac{\partial}{\partial t}[\rho \dot{z}'] &= -\left[c^2 \frac{\partial \rho}{\partial z'} + \rho \frac{\partial \mathrm{V}}{\partial z'}\right]. \end{aligned}\right\} \quad . \quad . \quad (11)$$

The equation of continuity is given as follows. The amount of material flowing into the volume is

$$-[\rho_1 \dot{x}] = -[\rho_1(\dot{x}_1' + u)],$$

while that flowing out is $(\rho_2(\dot{x}_2' + u))$.

Hence the total increase is given by

$$-\left[\left[+u\frac{\partial\rho}{\partial x'}+\frac{\partial}{\partial x'}(\rho\dot{x}')\right]+\frac{\partial}{\partial y'}(\rho\dot{y}')+\frac{\partial}{\partial z'}(\rho\dot{z}')\right]dx'\,dy'\,dz'.$$

This must be equated to the rate of change of ρ in the volume ; hence the continuity equation becomes

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x'}(\rho\dot{x}')+\frac{\partial}{\partial y'}(\rho\dot{y}')+\frac{\partial}{\partial z'}(\rho\dot{z}')+u\frac{\partial\rho}{\partial x'}=0. \quad (12)$$

Combining (12) with (11), there results

$$\frac{\partial^2\rho}{\partial t^2}=c^2\nabla'^2\rho+\mathrm{div}\,[\rho\nabla' \mathrm{V}], \quad . \quad . \quad . \quad (13)$$

where the primes indicate that the operations are to be performed with respect to the new variables x', y', z'.

Hence the equations of motion are invariant to a change from one set of axes to another moving with a constant velocity with respect to the first set of axes, and therefore the relativity principle is satisfied. Any experiment (such as the Michelson-Morley experiment) which attempts to measure the velocity of the earth with respect to the medium would yield a negative result according to this theory.

3. *The Distribution of Matter in Space.*

The Newtonian mechanical system described above results in a distribution of matter in space and time which in some respects is quite similar to that given by the general relativity theory. In other respects it is different, since it allows a periodic solution. Assuming a spherical universe, equation (10) reduces to the form

$$\frac{\partial^2\rho}{\partial t^2}=c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\rho}{\partial r}+k'\rho\int_0^r\lambda^2\rho d\lambda\right]\right], \quad . \quad (14)$$

where $k'=\dfrac{4\pi k}{c^2}$, k being the gravitational constant. This equation follows from (10), since

$$\nabla \mathrm{V}=\frac{\partial \mathrm{V}}{\partial r}=\frac{4\pi k}{r^2}\int_0^r\lambda^2\rho d\lambda. \quad . \quad . \quad . \quad (15)$$

Let us consider first the static solutions for the distribution of matter in the universe. These may be obtained by placing the left side of the equation equal to zero, resulting in the equation

$$r^2\frac{d\rho}{dr}+k'\rho\int_0^r\lambda^2\rho d\lambda=0. \quad . \quad . \quad . \quad . \quad (16)$$

This equation is reducible to a differential equation of the second order, and hence has two particular solutions. One of these is

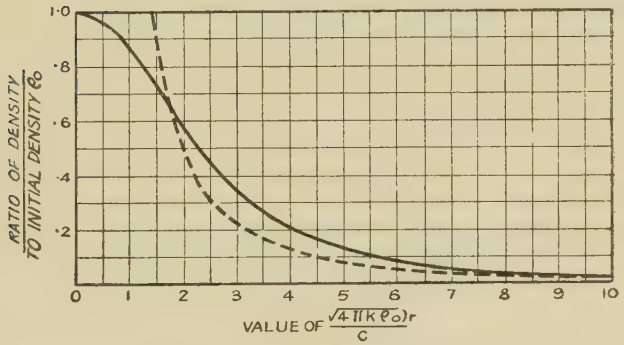
$$\rho = \frac{2}{k' r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

but this solution can be neglected when considered as a physical solution, since it results in an infinite potential energy at every point.

The second solution is difficult to obtain since the equation is a non-linear equation. For small values of r a rapidly converging power series is obtained from equation (16), the result being

$$\rho = \rho_0 \left[1 - \frac{k' \rho_0 r^2}{3!} + \frac{(8/3) k'^2 \rho_0^2 r^4}{5!} - \frac{\left(\frac{122}{9}\right) k'^3 \rho_0^3 r^6}{7!} + \frac{\left(\frac{5032}{45}\right) k'^4 \rho_0^4 r^8}{9!} - \frac{\left(\frac{181,864}{135}\right) k'^5 \rho_0^5 r^{10}}{11!} + \ldots \right]. \quad . \quad . \quad (18)$$

Fig. 1.



Stable density-distance distributions.

A plot of this equation is shown on fig. 1. No inverse power solution appears to exist. However, the solution of the equation for large values of r can be obtained by numerical and graphical methods, and it can be shown that for large values of r the solution approaches the first solution given by equation (17). We find then that if matter is distributed in accordance with either equation (17) or (18) it will be in equilibrium and will not change with time.

Both of these distributions require an infinite amount of matter for the whole universe. However, the space-gradients of the gravitational potential are finite at every point. The potential energy at any point of the medium is given by

$$W = c^2\rho + 4\pi k \int_0^r \frac{\rho}{\lambda^2} \int_0^\lambda \zeta^2 \rho \, d\zeta \, d\lambda \ldots \quad (19)$$

The first distribution given by (17) results in an infinite potential energy at every point, while the second one gives

$$W = c^2\rho_0 \ldots \quad (20)$$

at every point of the medium.

It is of interest to find whether the distribution of matter shown by equation (18) is a stable distribution or whether a slight change from the equilibrium position will cause a change to some other distribution. Suppose that we alter the distribution of matter from the equilibrium position given by (18) by replacing every value of r by (ar). For $a > 1$ the shape of the distribution will remain the same, but will be contracted in as though the unit of length had been decreased slightly. The equilibrium position for this distribution of matter will be reached if the density of matter at the centre of the coordinate system decreases and a general flowing of matter occurs in an outward direction. If a is less than one the reverse condition of affairs should take place.

If equation (18), with all r's changed to (ar)'s is substituted in (19), we find that

$$W = \frac{c^2}{a^2}[\rho_0 + (a^2 - 1)\rho] \ldots \quad (21)$$

The force causing a redistribution of matter is

$$F = -\frac{\partial W}{\partial r} = \frac{c^2}{a^2}(1 - a^2)\frac{\partial \rho}{\partial r} \ldots \quad (22)$$

From the differential equation (16) we see that $\frac{\partial \rho}{\partial r}$ is always negative for all values of r. Hence if $a^2 > 1$ the force applied along the radius is outwardly directed and tends to redistribute the matter towards the equilibrium position. Similarly, if $a^2 < 1$ the force is directed towards the centre and again tends to redistribute the matter towards its equilibrium position.

We can conclude then that the equilibrium distribution of matter given by equation (18) is a stable one, for if the distribution is slightly changed from the stable position

forces are set up which tend to redistribute the matter towards the equilibrium position.

4. *The Time Variation of a Finite Amount of Matter.*

According to the most recent developments of the relativity theory the conception is general that matter was originally distributed uniformly over a finite space and that it is now expanding at an ever increasing rate which will eventually approach the velocity of light. It is a matter of some interest to find what sort of a time variation of matter the present theory gives for initially finite distributions of matter. To simplify the calculation it is assumed that matter is initially distributed according to the equation

$$\rho=\rho_0 e^{-\frac{k'\rho_0 r^2}{6}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

For small values of r this distribution approaches the stable distribution given by equation (18). The total amount of matter in the universe according to this distribution is

$$M=4\pi\rho_0\int_0^{\infty}\lambda^2 e^{-\frac{k'\rho_0\lambda^2}{6}}d\lambda=\frac{(3/2)^{3/2}c^3}{k^{3/2}\rho_0^{1/2}} \quad . \quad . \quad . \quad (24)$$

The distribution will proceed towards the position of equilibrium, which from equation (18) will be $\rho_0=0$ for any finite distribution of mass. The initial stages of the motion, which is governed by equation (14), will be given by the series

$$\rho=\rho_0\Big[f_0(r)+\frac{t^2}{2!}f_1(r)+\frac{t^4}{4!}f_2(r)+\frac{t^6}{6!}f_3(r)+\ldots\Big]. \quad (25)$$

Substituting this equation in (14), we find that it is a solution if

$$f_1(r)=c^2\Big[\frac{1}{r^2}\frac{d}{dr}\Big[r^2\frac{df_0(r)}{dr}+k'\rho_0 f_0(r)\int_0^r\lambda^2 f_0(\lambda)d\lambda\Big]\Big],$$

$$f_2(r)=c^2\Big[\frac{1}{r^2}\frac{d}{dr}\Big[r^2\frac{df_1(r)}{dr}+k'\rho_0\Big[f_0(r)\int_0^r\lambda^2 f_1(\lambda)d\lambda+f_1(r)\int_0^r\lambda^2 f_0(\lambda)d\lambda\Big]\Big]\Big],$$

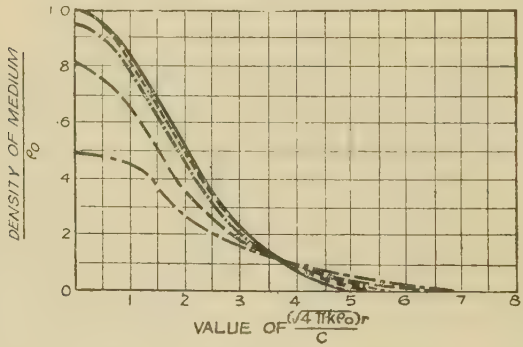
$$f_3(r)=c^2\Big[\frac{1}{r^2}\frac{d}{dr}\Big[r^2\frac{df_2(r)}{dr}+k'\rho_0\Big[f_0(r)\int_0^r\lambda^2 f_2(\lambda)d\lambda+6f_1(r)\int_0^r\lambda^2 f_1(\lambda)d\lambda+f_2(r)\int_0^r\lambda^2 f_0(\lambda)d\lambda\Big]\Big]\Big], \quad . \quad . \quad . \quad (26)$$

. . . . . .

and provided the velocity at any point of the medium is zero when $t=0$. Hence, if the initial distribution $f_0(r)$ is given, the distribution at any time can be calculated by (25) and (26).

Fig. 2 shows how the density of the medium varies with time for the initial stages of the motion. At the centre the initial motion is quite small until the elapsed time exceeds $\sqrt{4\pi k\rho_0}\,t=1/2$. The decrease in density near the centre of the universe will be approximately proportional to the square of the time for small values of t. Near the edges of the distribution the integral term on the right side of (14) can

Fig. 2.



A plot of the time-space distribution of matter for an initial distribution given by the equation $\rho=\rho_0 e^{\frac{-k'\rho_0 r^2}{6}}$.

———, initial distribution.

- - - - -, distribution when $\sqrt{4\pi k\rho_0}\,t=\cdot5$

—.—., distribution when $\sqrt{4\pi k\rho_0}\,t=1$.

— — —, distribution when $\sqrt{4\pi k\rho_0}\,t=1\cdot5$.

— - —, distribution when $\sqrt{4\pi k\rho_0}\,t=2\cdot0$.

be neglected, and here the motion is governed by the equation

$$\frac{\partial^2\rho}{\partial t^2}=c^2\nabla^2\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Hence, here the matter is propagated away from the centre with the velocity of light.

The final stages of the motion are evident from an inspection of the differential equation (14). As the density of matter decreases at any point the integral term of (14) decreases faster than the differential term in r, since the square of the density enters into the integral term. Hence, as the density

becomes less this term becomes smaller and the equation of motion approaches (27). As the density becomes smaller the velocity of propagation away from the centre becomes larger, and finally approaches the velocity of light.

We see, then, that the present theory gives much the same answer as the general relativity theory for a finite distribution of matter, namely, that it is unstable and will spread out from its initial distribution, at first slowly, and then finally with a velocity approaching the velocity of light.

5. *A Periodic Solution for the Distribution of Matter in the Universe.*

If the stipulation that the amount of matter in the universe must be finite is removed, it is possible to obtain a periodic solution. If the density of matter at creation were distributed in the equilibrium solution given by equation (18) no motion would take place thereafter. However, if it is distributed according to any other form a motion takes place which tends to redistribute the matter towards the equilibrium position. Since the system considered here is a non-dissipative one, the matter will not remain at rest when it approaches equilibrium, but by virtue of its kinetic energy will be carried beyond that position, and will oscillate back and forth around the equilibrium position. Hence there is reason to look for a periodic solution of equation (14).

In fact we find that (14) is satisfied by a series of the kind

$$\begin{aligned}\rho=\rho_0\big[&(1+a_0\sin\sqrt{4\pi k\rho_0}t)\\&+(a_1+b_1\sin\sqrt{4\pi k\rho_0}t+c_1\sin^2\sqrt{4\pi k\rho_0}t)r^2\\&+(a_2+b_2\sin\sqrt{4\pi k\rho_0}t+c_2\sin^2\sqrt{4\pi k\rho_0}t\\&\qquad+d_2\sin^3\sqrt{4\pi k\rho_0}t)r^4+\ldots\big]. \quad . \quad . \quad . \quad (28)\end{aligned}$$

Substituting in (14), we find that the above series satisfies the equation, and find that the first few coefficients are

$$\begin{aligned}\rho=\rho_0\Big[&(1+a_0\sin pt)-\frac{k'\rho_0 r^2}{3!}\cdot[1+3a_0\sin pt+a_0{}^2\sin^2 pt]\\&+\frac{8}{3}\frac{k'^2\rho_0{}^2r^4}{5!}\Big[\Big(1-\frac{3}{4}a_0{}^2\Big)+\frac{41}{8}a_0\sin pt\\&\qquad+\frac{11}{2}a_0{}^2\sin^2 pt+a_0{}^3\sin^3 pt\Big]+\ldots\ldots\Big],\end{aligned}$$

$$. \quad . \quad . \quad (29)$$

where p is substituted for the expression

$$p = \sqrt{4\pi k\rho_0}$$

and a_0 is a constant less than unity.

Hence it is evident that there exists a periodic solution in which ρ will vary from one extreme to the other with a period T equal to

$$\sqrt{4\pi k\rho_0}\,\mathrm{T} = 2\pi \quad \text{or} \quad \mathrm{T} = \frac{\sqrt{\pi}}{\sqrt{k\rho_0}} \ldots \quad (30)$$

From an experimental point of view the most important question is the velocity of recession or approach of one part of the medium towards another. This velocity can be calculated directly from (8), which for spherical coordinates becomes

$$\frac{\partial}{\partial t}(\rho\dot{r}) = -\left[c^2\frac{\partial\rho}{\partial r} + \frac{4\pi k\rho}{r^2}\int_0^r \lambda^2\rho\,d\lambda\right] \ldots \quad (31)$$

Using the value of ρ given by (29), we find that

$$\dot{r} = \frac{\dfrac{a_0 rp}{3}\left[\cos pt - \dfrac{3k'\rho_0 r^2}{10}\left[\cos pt + \dfrac{a_0}{3}\sin 2pt\right] + \dfrac{41k'^2\rho_0^2 r^4}{5!\times 7}\left[\left(1+\dfrac{6}{41}a_0^2\right)\cos pt + \dfrac{44}{41}a_0\sin 2pt - \dfrac{6}{41}a_0^2\cos 3pt\right] + \ldots\right]}{(1+a_0\sin pt) - \dfrac{k'\rho_0 r^2}{3!}\left[1+3a_0\sin pt + a_0^2\sin^2 pt\right] + \dfrac{8k'^2\rho_0^2 r^4}{3\times 5!}\left[\left(1-\dfrac{3}{4}a_0^2\right) + \dfrac{41}{8}a_0\sin pt + \dfrac{11}{2}a_0^2\sin^2 pt + a_0^3\sin^3 pt\right] + \ldots} \cdot \quad (32)$$

When r is small this reduces to

$$\frac{\dot{r}}{r} = \frac{\dfrac{a_0\sqrt{4\pi k\rho_0}}{3}\cos\sqrt{4\pi k\rho_0}\,t}{1+a_0\sin\sqrt{4\pi k\rho}\,pt}, \quad \ldots \quad (33)$$

which has for a maximum value

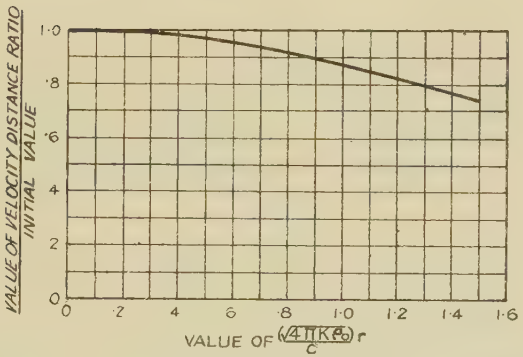
$$\frac{\dot{r}}{r} = \frac{a_0\sqrt{4\pi k\rho_0}}{3\sqrt{1-a_0^2}} \ldots \quad (34)$$

In order to agree with experiment the velocity of recession $\dot{r}$ should be capable of giving the value found by Hubble—$\left(\frac{\dot{r}}{r}\right) = 528$ kilometres per second per mega parsec,—and should not deviate appreciably from a proportionality to the radius for the farthest observable galaxy (140 million light-years). Some values for the minimum and maximum permissible densities are obtainable from these two considerations.

The minimum density may be calculated as follows:—According to (34), and assigning a reasonably large value to a_0, $\left(\frac{1}{\sqrt{2}}\right)$, the maximum velocity divided by distance is

$$\frac{\dot{r}}{r} = \frac{\sqrt{4\pi k\rho_0}}{3}.$$

Fig. 3.



A plot of the velocity-distance ratio as a function of the distance.

The value of Hubble's constant is $1{\cdot}72 \times 10^{-17}$. Hence the minimum value of ρ_0 which will meet this condition is given by

$$\frac{\sqrt{4\pi k\rho_0}}{3} = 1{\cdot}72 \times 10^{-17} \quad \text{or} \quad \rho_0 = 3{\cdot}2 \times 10^{-27}.$$

The maximum value of ρ_0 is given by the fact that the velocity-distance ratio does not remain a constant, according to equation (32), if the distance becomes too great. When $t = 0$, which is a representative case, the velocity-distance ratio should be as shown by fig. 3. Assuming that a deviation of Hubble's constant from a constant value could be detected if it were as great as 13 per cent., this would limit the upper

value of ρ_0. The farthest distant galaxy which has been observed is 140 million light years away. Hence to solve for ρ_0 we have

$$\frac{\sqrt{4\pi k\rho_0}\, r}{c} = 1.$$

Inserting a value of $\frac{r}{c} = (1{\cdot}4 \times 10^8)\,(3{\cdot}15 \times 10^7) = 4{\cdot}41 \times 10^{15}$, we find that

$$\rho_0 = 6{\cdot}1 \times 10^{-26} = \text{maximum value of } \rho_0.$$

Hence, according to this theory the density ρ_0 cannot differ greatly from

$$\rho_0 = 10^{-26}.$$

This gives for the period T a value equal approximately to

$$T = 2 \times 10^9 \text{ years.}$$

The relatively high value of the density found by this theory as compared to that measured by Hubble ($1{\cdot}5 \times 10^{-31}$) is no reproach to this theory, since any measurements of the density will measure only the concentrated matter in the form of stars, and not the distributed matter assumed here. The ratio of distributed matter to concentrated matter should be about 10^5. Hence the presence of concentrated matter in such a small ratio will have little effect on the motion of the distributed matter.

Since the stars are the sources of light whose displacement towards the red end of the spectrum is observed, it will be noted that this theory tacitly assumes that the stars are carried along on the average in the periodic movement of the medium considered here. Since the ratio of distributed to concentrated matter is so high and the time of transit so long, this appears to be a reasonable assumption.

Note added in proof.—It is, however, not necessary to make this assumption, for the motion of a dense body in the medium is easily calculated. Suppose that we consider a body of mass m at a distance r from the centre of the spherical universe. Then this body will experience a gravitational pull towards the centre given by

$$F = -\frac{km\mathrm{M}}{r^2}, \text{ where } \mathrm{M} = 4\pi \int_0^r \lambda^2 \rho\, d\lambda. \quad . \quad . \quad (35)$$

This force must be equated to the mass times acceleration in order to determine its motion. We are interested only in

the average motion of the body caused by the vibration of the medium. Hence the acceleration is divided into two parts, one a periodic one caused by the motion of the medium, and the other a fixed acceleration necessary to keep the dense body at a distance r from the centre of the universe. Hence

$$m\left[\frac{d^2r}{dt^2}+\frac{d^2r_0}{dt^2}\right]=-\frac{kmM}{r^2}\quad\ldots\quad(36)$$

Substituting the equation for the density of the medium (29) in (35), separating out the constant from the time variable parts, and solving for $\dot{r}=\frac{dr}{dt}$, we find

$$\frac{\dot{r}}{r}=\frac{a_0\sqrt{4\pi k\rho_0}}{3}\left[\cos pt-\frac{3k'\rho_0r^2}{10}\left(\cos pt+\frac{a_0}{3}\sin 2pt\right)\right.$$
$$+\frac{41k'^2\rho_0^2r^4}{5!\ 7}\left[\left(1+\frac{6}{41}a_0^2\right)\cos pt+\frac{44}{41}a_0\sin 2pt\right.$$
$$\left.\left.-\frac{6}{41}a_0^2\cos 3pt\right]+\ldots.\right].\quad\ldots\quad(37)$$

Comparing (37) with (32) it is seen that there is little difference between the motion of the medium and that of a dense body in the medium. For small values of r, the velocity of a dense body will be

$$\frac{\dot{r}}{r}=\frac{a_0\sqrt{4\pi k\rho_0}}{3}.\quad\ldots\quad(38)$$

For large values of r, the velocity distance falls off a little more rapidly than for the motion of the medium given by fig. 3.

The second equation, involving r_0, gives some idea as to the maximum distance a dense body can maintain itself from the centre of the universe. Considering a static universe, and substituting (18) in (35), there results

$$\frac{d^2r_0}{dt^2}=-\frac{4\pi k\rho_0r}{3}\left[1-\frac{k'\rho_0r^2}{10}+\frac{k'^2\rho_0^2r^4}{3\times5\times7}\cdots\right].\quad(39)$$

There are two methods for calculating the distance, a dense body can maintain itself from the centre of the universe. We can assume that it is traversing a circular orbit, or we can assume that it starts from the centre of the universe with a given velocity. There is no theoretical reason why a dense body should not have a velocity with respect to other bodies greater than the velocity of light. However, the process discussed above indicates that the dense bodies are

given their velocities by motions of the medium, and hence will not exceed a value c. Either method of calculation, assuming a limiting velocity c, gives as a result for the maximum radius, approximately

$$r_m = \frac{c}{\sqrt{\pi k \rho_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Using the value of ρ_0 found above, the maximum radius becomes

$$r_m = 7 \times 10^8 \text{ light-years}, \quad . \quad . \quad . \quad . \quad (41)$$

or about 5 times the farthest distance observed astronomically. Taking account of the periodic variation of the medium, *i. e.*, by integrating equation (37) with respect to t, this maximum distance becomes approximately

$$r_m = \frac{c}{\sqrt{\pi k \rho_0}} \left[1 + \frac{a_0}{3} \sin \sqrt{4\pi k \rho_0}\, t \right] . \quad . \quad . \quad (42)$$

These values set a limit to the observable universe.

Finally, we can conclude that the above outlined theory results in a cosmological system which does not disagree with any measured result, and does give a time duration for the universe large enough to agree with other methods of calculation, since a periodic solution results.

New York City.
Feb. 7, 1932.

XLIII. *The Photoionization of the Vapours of Anthracene and Diphenylamine.* *By* E. C. Allberry, *B.A., Scholar of Trinity College, Dublin* *.

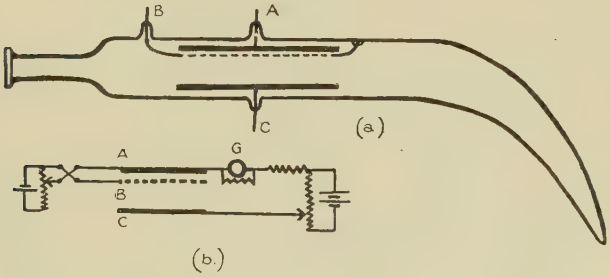
Stark † examined the photoionization of anthracene, diphenylamine, and other organic vapours by the increased conductivity produced in the vapour on illumination by a quartz mercury lamp. (All the vapours possess absorption bands in the near ultra-violet.) He used a quartz tube, heated by a Bunsen burner or a coil. With increased pressure of the vapour the current increased to a maximum, and then decreased. He concluded there was photoionization of the vapour, attributing the decrease of current from the maximum with increasing pressure to absorption near the

* Communicated by Prof. R. W. Ditchburn.
† J. Stark, *Phys. Zeits.* x. p. 614 (1909).

boundaries. With two volts across the electrodes he obtained currents up to 5×10^{-9} amp. (in the case of anthracene), and with twelve volts, up to 10^{-8} amp., at the pressure which gave the maximum current. Stark states that a defect of his experiments is the lack of quantitative measurement of the pressure; he estimated by observing a small discharge in the tube that the pressure at which the current was a maximum was 50 mm. of mercury. Gudden * suggested that Stark's observed currents may have been due to ultra-violet fluorescent light from the vapour acting photo-electrically on the nickel steel electrodes.

The writer tried to find the photoionization of anthracene vapour by the increase of space-charge limited currents when positive ions are present, a method used by Foote and Möhler † in the case of cæsium vapour. At a temperature

Fig. 1.



of 100° C., when the vapour pressure of anthracene is $6{\cdot}7 \times 10^{-2}$ mm. of mercury, no change in the space-charge limited current was observed on illumination of the vapour with a quartz mercury lamp, although the electron current was $1{\cdot}2 \times 10^{-6}$ amp. and a change of 3×10^{-9} amp. in the electron current could have been observed. This means that a positive ion current of 10^{-12} amp. should have been detected. The experiment was not, however, considered satisfactory, as the filament was easily poisoned by the anthracene at higher temperatures, causing the emission to vary a great deal and preventing observations with higher vapour pressures.

The tube with three electrodes shown in fig. 1 (*a*) was designed to distinguish the photoionization of the vapour postulated by Stark, and the indirect photo-electric effect

* B. Gudden, *Lichtelectrische Erscheinungen*, p. 231 (1928.)
† P. D. Foote and F. L. Möhler, Phys. Rev. xxvi. p. 195 (1925).

suggested by Gudden. It was intended to maintain A at about 3 volts negative to C, and to apply a small reversible potential (about 1 volt) between A and B. Then (I.) when B is positive to A, electrons can leave A, and (II.), when B is negative to A, no photo-electrons can leave A, but photo-electrons from the grid can reach A. In either case photo-ions produced in the gas are driven on to A. Thus the positive current in case (I.) should consist of photo-ions plus the photo-electric current from A, and in case (II.) of photo-ions minus a smaller number of photo-electrons from the grid. In this way the two effects can be clearly distinguished, For preliminary tests B was connected to A. With this arrangement the current to A consisted of the photo-ions plus some photo-electrons from A and B.

Apparatus and Experiment.

A pyrex tube was constructed as shown in fig. 1 (*a*). One end was drawn out into a Wood light-trap, at the other a fused quartz window 1·5 cm. in diameter was cemented on with Robbialac cycle enamel. This was done by grinding flat the end of a pyrex tube, clamping it against the quartz, and painting round. When dry, it was baked for 24 hours at 210° C. Two coats were given thus, and the resulting joint was such that the tube when evacuated and tested with a coil showed no discharge after 48 hours. It withstood 210° C., the highest temperature tried. A side tube served to introduce substances examined. The electrodes A and C were of nickel foil, $9 \times 1{\cdot}3$ cm. and 2·3 cm. apart. B was a grid of nickel wire, of 2 mm. mesh, 3 mm. from A. The circuit (fig. 1 (*b*)) consisted of a galvanometer with a 20,000 ohm shunt, a megohm resistance, and a potential divider giving potentials up to 10 volts. The galvanometer sensitivity was $1{\cdot}60 \times 10^{-10}$ amp./mm. when shunted with the above resistance.

The method used with both substances was the same. The tube was baked out at 210° C., the substance put in, and the tube sealed off from the pump when no discharge of any kind could be produced by a small coil. It was then heated in an electric oven at temperatures up to 200° C., at which temperature the vapour pressure as estimated by extrapolation from the figures given in the International Critical Tables are 36 mm. of mercury for anthracene and 57 mm. for diphenylamine. In both cases there was a "dark" current of $1{\cdot}44 \times 10^{-9}$ amp. at 200°; this is of the same order as the "dark" current with the tube empty, and was probably due

to the conductivity of the glass. On illumination of the vapour by a parallel beam of light from an iron arc there was no change in the current, even on removal of the stops which shielded the electrodes from direct illumination. A was connected to B and the potential across AC was 10 volts. In the case of anthracene a quartz mercury lamp was used with the same results.

Direct tests showed that lines down to 1900 Å.U. were given by the arc and were transmitted through the lens and window used in the experinent.

The tube at room temperature was then connected to an electrometer with an earth leak. With the 10 volt potential, on illumination of the electrodes with the iron arc, a current of the order of 5×10^{-10} amp. (corresponding to a deflexion of about $\frac{1}{2}$ mm. on the galvanometer) was obtained; as the intensity of this illumination was presumably greater than that possible by fluorescence of the vapour, it is not likely that Gudden's explanation of Stark's results is correct.

It seemed just possible that, owing to the irregular heating of Stark's tube, the currents were due to the photo-conductivity of a film of solid on the quartz. A direct experiment was made to see if there was a large conductivity of solid anthracene, but no current was detected across a film of the solid on a glass plate (using a potential of ten volts, and the same galvanometer) even on illumination by the iron arc.

The following is a possible explanation of Stark's results. Owing to the uneven heating of his tube, films of solid anthracene may have been present on the electrodes. Fluorescent light from the vapour falling on these might cause a much larger emission of electrons than would be obtained from the metal itself. In support of this view it may be noted that Hughes * found a very large photo-electric effect with solid anthracene for wave-lengths shorter than about 2200 Å.U.

These experiments therefore show :—

(*a*) That Stark's currents were not due to genuine photo-ionization.

(*b*) That the explanations so far suggested are not correct.

The writer has been unable to find any satisfactory explanation.

It should be clearly understood that these experiments do not exclude the possibility of small photoionization effects

* A. Ll. Hughes, Phil. Mag. xxiv. p. 380 (1912).

in the vapours, but there is now no practical evidence and no theoretical reason for believing that such an effect exists.

The writer is indebted to Professor Ditchburn for invaluable suggestions and advice.

Summary.

These experiments were undertaken to test the results of Stark, who obtained currents of the order of 10^{-10}–10^{-8} amp. on illuminating the vapours of anthracene, etc. In the present case the vapour was contained in a pyrex tube with a quartz window and nickel electrodes. There was a fairly small steady "dark" current; no change was observed in this on illuminating the vapour with an iron arc, using a galvanometer which would show definitely currents of 10^{-10} amp Currents just less than this were observed on illuminating the electrodes at room temperature, using an electrometer with an earth leak. It is concluded that the currents observed by Stark were not due to direct photoionization of the vapour. Other possible explanations of his results are examined.

XLIV. *A Wedge Photometer for Daylight Factor Measurements.* *By* D. R. Barber, *B.Sc., A.Inst.P., Washington Singer Laboratories, Exeter* *.

IN the measurement of daylight illumination with brightness photometers incorporating a metallic filament lamp as light-source, it is necessary to interpose a suitable colour filter between source and screen so as to obtain, at the comparison surface, a quality of illumination corresponding to the incident daylight. Even, in such circumstances, considerable difficulty may be experienced in obtaining a perfect match, chiefly owing to the very variable quality of the skylight illuminating the photometer aperture. Consequently, in any direct method for the comparison of the intensity of illumination within a building with that due to an unobstructed hemisphere of uniformly radiated skylight, *i.e.*, daylight factor measurement, it is desirable that some direct visual means shall be employed which is independent of any artificial light-source. Two instruments fulfilling this desideratum have been described by Taylor [1], but, in each

* Communicated by Prof. F. H. Newman, D.Sc., F.Inst P.

case, the daylight factor is expressed in terms of a rather variable quantity—the "sill ratio."

Some experiments upon new methods applicable to daylight factor measurement have been recently made by the author in the course of an investigation into the distribution of natural illumination within these laboratories. It is therefore thought that a detailed description of an experimental instrument which has been used for this purpose will be of interest. The theory of the instrument is outlined, and its performance is compared with that of a standard type of brightness photometer—the Holophane Lumeter.

Theoretical Treatment.

It has long been realized that the resolving power of the eye (expressed in terms of visual acuity) and its ability to discriminate differences (expressed as contrast sensibility) are both primarily dependent upon the intrinsic brightness of the object viewed. According to the theory of Weber, later elaborated by Fechner, the least difference in the brightness of an illuminated object for the sensation of contrast to be just perceptible bears a fixed ratio to the intrinsic intensity of illumination. Thus, if ΔI represents this critical difference, and I is the intrinsic intensity, then

$$\frac{\Delta I}{I} = c, \text{ where } c \text{ is Fechner's constant.}$$

The quantity ΔI is termed the differential threshold for the eye. The first comprehensive investigation of the above relationship was made by König [2], whose results confirmed the law, but Parsons [3] and later Hecht [4] both conclude that Weber's law is not followed rigidly for either very low or very high intensities, and, in consequence, is only an approximation. Recently Macdonald and Allen [5] have obtained experimental verification of the law over a wide range of intensities, but they find a series of values must be assigned to the constant, c, in order to validate the relation over such a range. They further consider that Weber's conclusions are only strictly applicable to problems which involve a change in the intensity of a single stimulus, and not to the contrast sensations evoked by individual stimuli of differing magnitude emitted by adjacent sources.

In the light of present experimental evidence it is, however, reasonable to assume the applicability of the law of contrast intensities to the restricted conditions of the arrangement here described where we are dealing with a large number of very small contrasted elements. Thus, if the eye

of an observer is directed towards an illuminated screen consisting of alternating areas of neutral grey and white (fig. 1 *b*), there will be a certain critical value for the contrast intensity at the eye for which the pattern will apparently just disappear, a uniform grey hue resulting. This critical value has already been defined as the differential threshold. In the experimental arrangement here used, a neutral, circular, absorption wedge is interposed between the test-screen, uniformly illuminated at a fixed brightness value, and the observer's eye, the differential threshold being obtained by rotating the wedge until the screen pattern just vanishes. This value may then be related to the intrinsic intensity at the screen in terms of the angle through which the wedge has been turned. The wedge is graduated through the range 0°–360° in the direction of increasing absorption. The magnitude of the differential threshold, ΔI, for white light, will be determined by the contrast sensibility of the observing eye. Houston and Shearer [6] state that this power of discriminating brightness differences, for any fixed intrinsic intensity, will fluctuate for the same observer over a period of time. While Blanchard [7] also points out that the least brightness difference detectable will depend, not only upon the intrinsic intensities of the contrasting fields, but also upon their contour and extent, the previous adaptation of the eye, and the duration of exposure to the field observed. Consequently, any contrast method must be such that the above factors are controllable, if accuracy is to be achieved.

Suppose, that for a given angular displacement, θ, the intensity of the light incident upon the wedge is I, while the mean intensity after passing through the absorbing medium is I_θ. Let the density of the absorbing deposit be Δ_θ defined as $\log_e\left(\frac{I}{I_\theta}\right)$.

The relationship existing between the rotation, θ, and the intensities, I and I_θ, of the incident and transmitted light is found, by direct calibration, to be exponential in form (fig. 2) and may therefore be expressed by the equation

$$I_\theta = Ie^{-\alpha\theta-\beta}, \text{ whence } \frac{I}{I_\theta} = e^{\alpha\theta+\beta}, \quad . \quad . \quad . \quad \text{(i.)}$$

or, in logarithmic form,

$$\log_e\left(\frac{I}{I_\theta}\right) = \alpha\theta + \beta = \Delta_\theta, \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

where α and β are both constants.

Thus the density of the wedge deposit is proportional to the angle of rotation. If, now, θ_c represents the rotation necessary to reduce the intensity, I, at the screen to an apparent intensity, I_c, for which the contrast ratio just equals the differential threshold for the observer, the screen pattern will no longer be resolvable, and we then have from equation (ii.)

$$\log\left(\frac{I}{I_c}\right)=\alpha\theta_c+\beta, \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

where again α and β are constants ;

whence $$\log_e I=\alpha\theta_c+\beta+\log_e I_c. \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

Now, for a particular combination of wedge and screen, the intensity term, $\log_e I_c$, will have a constant value dependent upon the contrast sensibility of the eye, and equation (iv.) may therefore be simplified to the final form

$$\log_{10} I=a\theta_c+b, \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}$$

where a and b are constants, the logarithms having been converted to the common base for convenience of calibration. The actual calibration curve shows that, for the range of intensities available, relation (v.) holds, the values of the constants being $a=+0{\cdot}025$, and $b=-2{\cdot}175$ for the particular wedge and screen used. The equation to the calibration curve is therefore

$$\log_{10} I=0{\cdot}025\theta-2{\cdot}175, \quad . \quad . \quad . \quad . \quad \text{(vi.)}$$

and the value of I, the required intensity of illumination at the screen, may be calculated directly in terms of the wedge scale-reading, θ.

Experimental.

The apparatus used consisted of a modified Zeiss Ikon "Diaphot" photographic exposure meter, and a specially prepared test-screen. Both are shown diagrammatically in figs. 1*a* and 1*b*. This particular meter was chosen for adaptation as it was fitted with a suitably graded wedge filter covering a convenient density range for existing requirements. The only alteration was the addition of a circular scale, S, which was attached to the metal frame, M, of the meter, and a small metal index, I, soldered to the outer rotating ring, R. The scale, $1\frac{3}{4}$ inch diameter, was prepared by photographic reduction from a paper galvanometer scale divided into degrees, 0°–360°. To accommodate this index the handle support, H, was slotted as indicated in the figure. It was deemed unnecessary to remove the light

blue filter, F, interposed between wedge, W, and eyepiece, E, which is a standard fitting to the "Diaphot" meter. The

Fig. 1 *a*.

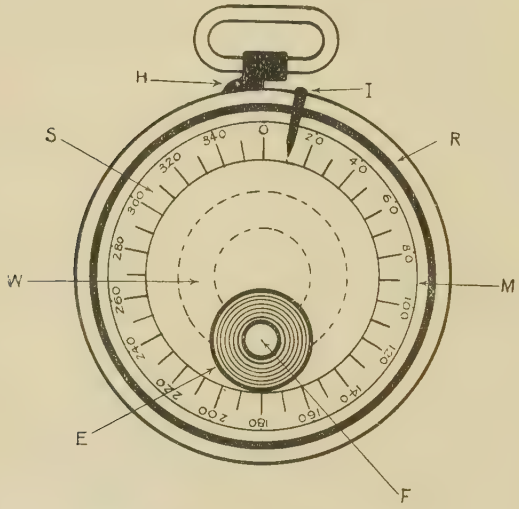
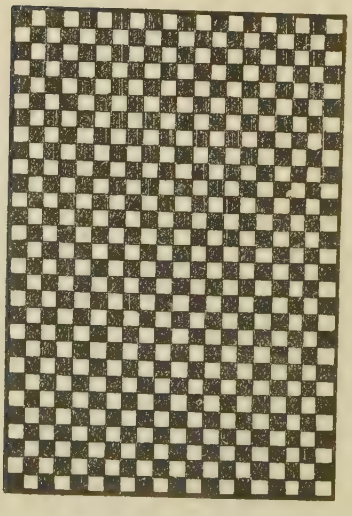


Fig. 1 *b*.



test-screen (fig. 1 *b*) was also prepared photographically, the method adopted affording a convenient means of obtaining a number of screens possessing varying degrees of contrast.

A negative was first prepared by direct photography of a black and white check pattern sheet, and from this a series of prints were made, the exposure being increased by suitable steps so as to yield a contrast pattern in which the dark areas ranged from very light to very dark grey. Each screen was then tested in turn over the range of light-intensities available and the most suitable contrast selected for subsequent calibration. The correct choice of contrast is important, since the use, over a wide range, of a screen suitable only for either high or low intensity values, will adversely affect the degree of accuracy with which the wedge may be adjusted. The size of the screen pattern is also rather critical because the screen is viewed at a fixed distance, normally one metre, from the wedge. It therefore follows that the pattern must have such dimensions that, when viewed at the standard distance, the detail is readily distinguishable at zero setting of the wedge when the screen is illuminated at the minimum intensity to be subsequently measured. Conversely, an element whose area is such that it will subtend a large angle at the eye is not admissible from theoretical considerations. The screen finally selected possessed medium contrast, the alternate squares each having an area of 9 sq. mm., the total area viewed being 56·25 sq. cm. The angle which each element thus subtended at the eye was 10′ when viewed from a distance of one metre. The screen was printed on platino-matt bromide paper, and mounted on glass. A quarter-plate single metal slide protected its surface from injury when the photometer was not in use.

Calibration of the Photometer.

(a) *Wedge.*

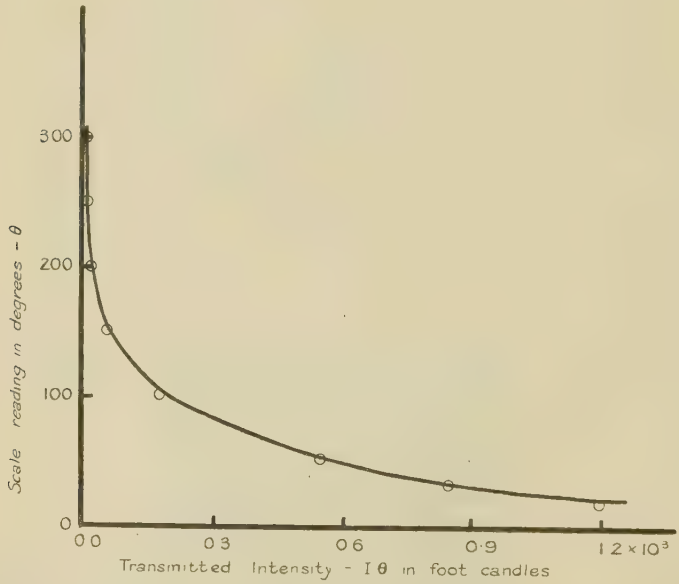
The wedge filter was calibrated photometrically in terms of the angle of rotation, and the intensity of the transmitted light. For this purpose the meter was mounted in a vertical plane so as to completely cover the aperture of a Model B Holophane Lumeter. A 220 v. 60 w. pearl Mazda lamp furnished a steady source of light. Calibration was effected by turning the wedge through 10° for each observation, and the required intensities at the face and rear of the wedge were obtained directly from the Lumeter scale-readings. The curve connecting the intensity of the transmitted light is reproduced in fig. 2.

(b) *Screen.*

Before the arrangement described could be used for the determination of daylight factors it was necessary to correlate

the intrinsic intensity of the light incident upon the screen itself, and the rotation of the wedge yielding the required threshold at the eye. A 220 v. 500 w. Osram lamp was set up at one end of a dark room, and the screen mounted in a vertical position facing the light-source. By sliding the screen holder backwards and forwards along the bench the screen could be illuminated with light of a calculated intensity, the distance from screen to lamp being measured for each observation made. In each of eight different positions the screen was viewed as nearly normally as was possible in

Fig. 2.



order to obviate the effect of obliquity, and six readings of the wedge were obtained for which the pattern just became resolvable. At first it was not possible to obtain highly concordant readings but, after some observational practice, values were obtained agreeing to within 3°. Fig. 3 shows the graph of $\log I$ plotted against θ, and the result agrees with that expected on theoretical grounds. The direct calibration covered an intensity range of 0·5–150 foot candles.

The effect of the apparent size of the screen pattern was also investigated at the same time. Comparative values of

Fig. 3.

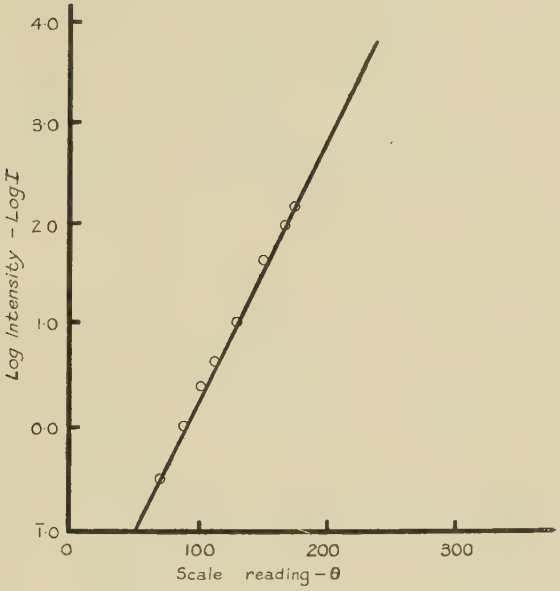
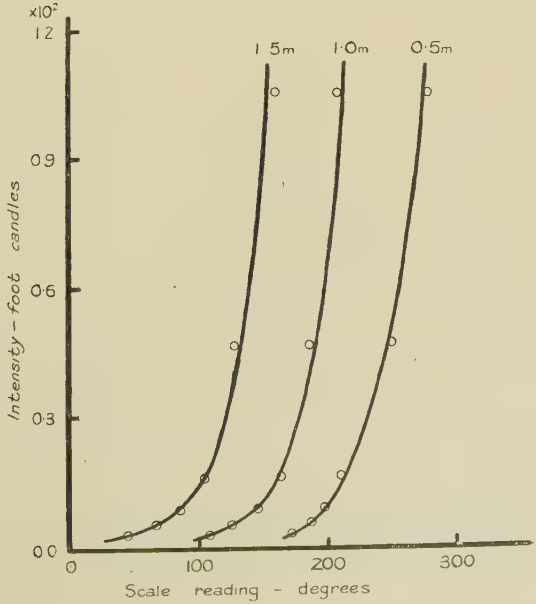

4·0
3·0
2·0
1·0
0·0
$\bar{1}$·0
Log Intensity – Log I
0
100
200
300
Scale reading – θ

Fig. 4.


x10²
1·2
0·9
0·6
0·3
0·0
Intensity – foot candles
1·5m
1·0m
0·5m
0
100
200
300
Scale reading – degrees

the wedge scale-reading corresponding to six measured intensities of illumination of the screen were obtained by viewing the latter from distances of 0·5, 1·0, and 1·5 metres. These values corresponed to angles of 21′, 10′, and 7′ subtended at the observer's eye. Results are shown graphically in fig. 4 and, as might be expected, it will be seen that the smaller pattern yields a calibration curve most suitable for measurements in bright light, whilst for weak illumination it would be desirable to make use of a slightly coarser screen.

Daylight Factor Measurements.

The normal range of intensities obtaining in daylight factor measurements lies within the range 0·25–2000 ft. candles, for although the diffuse daylight value on a clear day with bright sunlight may rise to as high a figure as 10,000 ft. candles, the condition recognized as being desirable for reliable results, viz., a dull grey sky, limits the range as stated above. Even so, it is necessary to extrapolate from the calibration curve values above 150 ft. candles. An inspection of the curve will indicate that, over the upper range, a variation of only one degree will introduce a large error in evaluating I. After preliminary practice it has, however, been found practicable to limit this error to a mean value of 10 per cent. when the wedge is being used over the upper range from 200°, the percentage error being proportionally reduced for lower values of θ. Table I. has been included to illustrate the closeness to which scale-readings actually agree when the photometer is used out of doors, *i. e.*, over this upper range. The figures have been selected from observations extending over a period and are divided into two groups :—(*A*) records of external daylight intensities obtained on dull days with overcast sky, and (*B*) records obtained with the test-screen exposed to direct sunlight, supplemented by readings in the shade.

The site chosen for the observations of the direct skylight intensity should conform to the conditions of uniform radiation from an unobstructed hemisphere of sky, but it is not readily possible to attain this. In the experiments here recorded the orientation of the site was such that a portion of the complete hemisphere was cut off by the ridge of the main roof, while in the opposite direction rising ground, fairly studded with trees, screened the true horizon line. In consequence, all the values for daylight factor which are quoted may be regarded as being at least 10 per cent. too high.

The test-screen was always used with its surface in a horizontal plane, and for the interior observations it was placed in the room, at the desired location, at a standard height of 3 ft. 6 in. above the floor level. Each observation normally consisted of a series of four separate scale settings occupying 30–45 seconds, the observer's eye having been dark adapted for a period of 30 seconds previous to every group. When observing at the higher intensities, *i. e.*, out of doors precautions were taken to minimise, as far as

TABLE I.

Group.	State of sky.	Position of screen.	θ.	I.
A.	V. dull with heavy cloud.	Exposed horizontally.	195 197 196	501 562 531
	Broken sky, heavy cloud.	Ditto.	208 211 209	1,047 1,245 1,122
B.	Clear sky, sun shining.	Exposed vertically.	264 267 264	26,610 31,620 26,610
	Ditto.	Exposed horizontally.	251 248 250	12,590 9,226 11,890
	Ditto.	Exposed vertically in shade.	201 199 202	708 631 750
	Ditto.	Exposed horizontally in shade.	217 217 219	1,778 1,778 1,995

possible, inaccuracy in the wedge setting due to the following causes:—(*a*) effect upon the contrast sensibility of the eye of scattered light entering the eye direct, (*b*) combined effect of induction and after-image when the screen was viewed through the most transparent portions of the wedge.

It was found possible to reduce the first by making use of an additional cupped eyepiece so as to more effectively screen the observing eye. The initial method employed in obtaining the critical angle of rotation involved turning the wedge in the direction of increasing absorption. This led to very pronounced eye fatigue, especially whenever the screen was brightly lit. As stated above, this was due, firstly, to

inductive dazzle in the initial stages of the setting; and secondly, to delayed retinal response when the intensity at the eye was approaching the required threshold value.

Both effects were subsequently eliminated almost entirely by adopting the reverse procedure, the wedge rotation commencing from the position of maximum absorption and continuing until the screen pattern was just resolvable. This procedure has consequently been adhered to throughout the work.

Table II. yields a direct comparison of the results obtained from the wedge method with corresponding values of daylight factor as calculated from readings obtained with a model B Holophane Lumeter.

TABLE II.

State of sky.		Very dull.		Broken clouds.		State of sky.		Very dull.		Broken clouds.	
Time of day.		M.	A.	M.	A.	Time of day.		M.	A.	M.	A.
		%	%	%	%			%	%	%	%
1 *a*	W	0·14	—	—	0·09	1 *f*	W	0·55	—	—	0·37
	L	0·16	—	—	0·06		L	0·44	—	—	0·50
1 *b*	W	0·22	—	—	0·10	2	W	—	3·02	3·50	—
	L	0·31	—	—	0·21		L	—	3·06	4·33	—
1 *c*	W	0·17	—	—	0·18	3	W	1·81	—	1·82	—
	L	0·17	—	—	0·19		L	1·90	—	1·74	—
1 *d*	W	0 37	—	—	0 50	4 *a*	W	0·33	—	—	0·11
	L	0·53	—	—	0·51		L	0·29	—	—	0·16
1 *e*	W	3·49	—	—	3·05	4 *b*	W	0·65	—	—	0·47
	L	3·43	—	—	2·82		L	0·61	—	—	0·58

The above figures have been selected at random from a series of periodic records of daylight factor for various rooms in the building. The index symbols in the first and fourth columns above refer to the room and the position in which the test was carried out. Reading along the horizontal row the prefixes W and L indicate readings obtained with wedge and Lumeter respectively. Results are included for morning (M) and afternoon (A) observations. It will be seen that the agreement in the values obtained by the two methods is consistently close.

SUMMARY.

1. The paper describes a method for the evaluation of daylight factor which is based upon the fact that a critical

intensity of illumination is required to enable a detailed object to be just resolved.

2. The theory of the method is outlined, and calibration data are given for an experimental photometer.

3. The performance of the instrument is compared with a standard Holophane Lumeter.

The author desires to express his indebtedness to Prof. F. H. Newman, D.Sc., who suggested this research, for his interest and advice throughout the investigation.

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XLV. *The Equilibrium between Matter and Radiant Energy.*

By J. A. C. Teegan, *M.Sc., A.Inst.P., M.R.I.A., Assistant in Physical Chemistry, University College, Cork* *.

To account for the enormous amount of energy radiated from the hot stars Jeans has suggested the possibility of the annihilation of matter by the coalescence of an electron and a proton, and their conversion into radiant energy of frequency ν, in accordance with Einstein's fundamental law :—

$$c^2m=h\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Substituting in this equation the values $c=3\times10^{10}$, $m=1{\cdot}66\times10^{-24}$, and $h=6{\cdot}5\times10^{-27}$, the frequency of the radiation corresponding to the annihilation of a single atom of hydrogen comes out as $\nu=2{\cdot}3\times10^{23}$, corresponding to a wave-length $\lambda=1{\cdot}3\times10^{-13}$ cm. The short wave limit of the so called "cosmic-ray" spectrum corresponds to radiation having exactly this wave-length, and the simplest interpretation of this fact is that these short waves originate out of either (1) the complete transformation of single free atoms

* Communicated by the Author.

of hydrogen into radiation or (2) the neutralization of a proton and electron in some heavy atom, of which the atomic weight is thereby reduced by unity.

Jeans has further suggested that such annihilation is only likely to occur in atoms which are not found on this earth, *i. e.*, in atoms of higher atomic weight than terrestrial atoms. He considers the possibility of such transformations occurring in the stars, citing as evidence the fact that the youngest stars are statistically the most massive, pointing to the fact that they lose much of their mass in the course of their lives.

Two types of annihilation are possible (1) : a spontaneous annihilation analogous to radioactivity taking place regardless of temperature and pressure, (2) an annihilation excited by high temperature of which the rate increases with rise of temperature.

In this paper an approximate method of investigating the temperatures at which such changes occur is indicated.

The fundamental process of (proton-electron) annihilation may be represented :

$$M_H + M_e = \frac{h\nu}{c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where M_H = mass of proton, M_e electronic mass. We might regard this process as an exothermic chemical transformation which takes place only at some very high degree of temperature (or other source of excitation). The reverse process,

$$\frac{h\nu}{c^2} = M_H + M_e,$$

being endothermic, will only proceed at some much higher temperature, at which it is possible to picture a condition of equilibrium existing in which the forward and back processes are proceeding simultaneously. In order to get some idea of the temperature at which such an equilibrium might exist, let us consider a closed space in which protons, electrons, and photons of frequency $2{\cdot}3 \times 10^{23}$ are in a state of equilibrium

$$M_H + M_e \underset{\leftarrow}{\rightarrow} \frac{h\nu}{c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This condition may be likened to a gas equilibrium of the type

$$N_2 + O_2 \underset{\leftarrow}{\rightarrow} 2NO. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In such a chemical system the equilibrium is displaced by a

rise of temperature in that direction in which the reaction takes place endothermically.

The displacement of the equilibrium by a change of temperature is expressed by Van't Hoff's isochore

$$\frac{\partial}{\partial T}\log_e K = -\frac{Q}{RT^2},$$

where K is the equilibrium constant, Q is the molecular heat of reaction, and R the gas constant.

Assuming Q is independent of temperature, this equation becomes on integration

$$\log_e K = \frac{Q}{RT} + \text{const. or } K = Ce^{\frac{Q}{RT}}.$$

From this equation it follows that if Q is negative, then K increases with rise in temperature, so that the equilibrium is displaced in the direction proceeding with absorption of heat.

To apply this equation to the equilibrium

$$M_\pi + M_\epsilon \underset{\leftarrow}{\rightarrow} \frac{h\nu}{c^2},$$

we must replace Q and R by their values for atomic dimensions, and we have (remembering that Q is negative)

$$K = Ce^{-\frac{q}{kT}}, \text{ where } q = \frac{Q}{N} \text{ and } k = \frac{R}{N} \quad . \quad . \quad . \quad (7)$$

It is obvious that q in this case is very large ($=h\nu=mc^2$), so that kT must be comparable to q if K is to have an appreciable magnitude. Thus, in order that the reverse reaction consisting of the condensation of radiation into matter may proceed to any appreciable extent, a very high temperature is required of the order given by

$$T = \frac{mc^2}{k} \doteqdot \frac{1{\cdot}66 \times 10^{-24} \times 9 \times 10^{20} \times 6 \times 10^{23}}{8{\cdot}4 \times 10^{8}}$$
$$\doteqdot 10^{12}\,\text{Å}.$$

The exponential term in equation (7) increases very rapidly with temperature T, so that the value of K will diminish correspondingly rapidly as the temperature falls below 10^{12} Å.

This result is in good agreement with the value obtained by Milne from an application of the Fermi-Dirac quantum mechanics to this equilibrium. It should be noted that this temperature is far in excess of that necessary for the direct

annihilation process of Jeans, and represents the temperature condition under which matter would be continually breaking up into radiation, and simultaneously the radiation would be condensing into matter. Milne has actually shown that the annihilation process is almost completely developed at a temperature of 10^{10} Å.

Chemical Department,
University College, Cork.
April 14, 1932.

XLVI. *Excitation of Continuous and Line Spectra in Helium**. *By* J. S. Townsend, *M.A., F.R.S., Wykeham Professor of Physics, Oxford, and* M. H. Pakkala, *B.A., Jesus College, Oxford* †.

1. THE investigations of discharges through gases in long tubes have shown that a consistent theory of several phenomena is obtained on the hypothesis that positive ions and electrons disappear by impacts on the surface of the tube, and that there is no appreciable decrease of conductivity due to recombination in the gas.

It follows from this hypothesis that in the uniform positive column of a discharge where the force is independent of the current there is a positive charge in the gas, so that positive ions and electrons disappear at the same rate from the gas by diffusion to the surface of the tube ‡.

At the surface of the tube the positive ions combine with electrons and return to the gas as neutral atoms or molecules; but in this process no appreciable radiation is emitted, as is seen by considering the Faraday dark space or the change in the luminous column due to an increase of pressure. Since the electric force is greater in the positive column than in the Faraday dark space, it would be unreasonable to assume that the rate of recombination is greater in the luminous column than in the dark space. Thus the light from the positive column cannot be attributed to recombination.

The explanation of the positive column and the Faraday dark space is quite simple if the radiation be attributed to

* This paper was read at a meeting of the Physical Society of London held at the University College, Nottingham, on the 11th June, 1932.

† Communicated by the Authors.

‡ 'Electricity in Gases,' Clarendon Press, Oxford (1915).

collisions of electrons with atoms or molecules of the gas. When the velocity of agitation of the electrons is very small the radiation is not appreciable, but there is a marked increase in the intensity of the radiation due to a comparatively small increase in the mean velocity of agitation of the electrons. In the Faraday dark space the electric force is small and the energy of the electrons is too small to excite radiation, but in the luminous positive column the force is greater than in the Faraday dark space, and the luminous glow is due to the increase in the energy of agitation of the electrons.

2. A similar explanation of the changes in the intensity of the light from discharges due to changes of the pressure of the gas is obtained by considering the electric force Z and the pressure, p, of the gas, since the kinetic energy of the electrons in a long uniform column depends only on the ratio Z/p.

This explanation of the change of intensity of the light in the positive column of a discharge was first given in the treatise on 'Electricity in Gases' (Section 274), where Graham's* experiments on the distribution of light and electric force in discharges through nitrogen were described.

These experiments were made with a tube 3·3 cm. in diameter, with two plane electrodes 20 cm. apart. With nitrogen at 0·95 mm. pressure and a current of $2{\cdot}46 \times 10^{-3}$ amp. through the gas there is a uniform luminous column extending from the positive electrode to a distance of 10 cm. along the tube. The force was 20 volts per cm. in the uniform column. In the Faraday dark space, which was 8 cm. long, the force was not uniform and the mean value was about 10 volts per cm. In this case the ratio Z/p is 21 in the luminous column and about 10·5 volts in the dark space.

With nitrogen at 2·5 mm. pressure and a current of $2{\cdot}39 \times 10^{-3}$ amp. there is a dark discharge throughout the tube except in the spaces about 2 or 3 mm. wide near the electrodes. There is a uniform force of 25 volts per cm. in the space, from 2 to 10 cm. from the positive electrode (where the positive column at lower pressures is visible); and in the place of the Faraday dark space the mean value of the force was 18 volts per cm. Thus the discharge is dark when the ratio Z/p is 10 in the space where a bright uniform column is obtained with larger values of the ratio Z/p. The change from the luminous to the dark discharge in this space, due to the change in

* W. P. Graham, Wied. *Ann.* lxiv. p. 49 (1898).

pressure from 0·95 mm. to 2·5 mm., may be attributed to the change in the velocity of agitation of the electrons corresponding to the change in the ratio Z/p from 21 to 10.

3. Recent experiments on discharges in helium and in neon have shown that in the uniform positive columns of the discharges the mean energy of agitation of the electrons is about three volts; but a small proportion of the electrons have comparatively large energies, and the ionization and the excitation of the line spectrum of the gas are due to collisions of electrons with atoms of the gas in which the energy of the electrons is much greater than the mean.

In the investigations which have been made to examine this theory the photoelectric effect of the lines in the visible spectrum was measured for a constant current in the gas at different pressures *. It was thus found that the intensity of any line and the mean energy of agitation of electrons diminished as the pressure was increased, but the change in the intensity of the light was much greater than the change in the mean energy of agitation. According to this theory the greater the change in the intensity of the light due to a small change in the mean energy of agitation of the electrons the greater the energy of the electrons that excite the light †.

4. In the spectra of the uniform columns of discharges in helium, as seen in a direct-vision spectroscope, there are dark spaces between the ordinary helium lines when the pressure of the gas is less than 2 or 3 mm. At higher pressures when the gas is contained in wide tubes, about 3 cm. in diameter, there is continuous light between the lines, and as the pressure is increased the intensity of the continuous spectrum increases in comparison with that of the line spectrum. Faint bands are also seen in the positive column at the higher pressures, but there is a wide space between the principal green line (λ 5016) and the yellow line (λ 5876), where there are no bands ‡. In this part of the spectrum of the uniform column the only noticeable effect of increasing the pressure is the increase in the

* J. S. Townsend and F. Llewellyn-Jones, Phil. Mag. xi. p. 679 (March 1931). xii. p. 815 (Oct. 1931); P. Johnson, Phil. Mag. xiii. p. 487 (Feb. 1932).

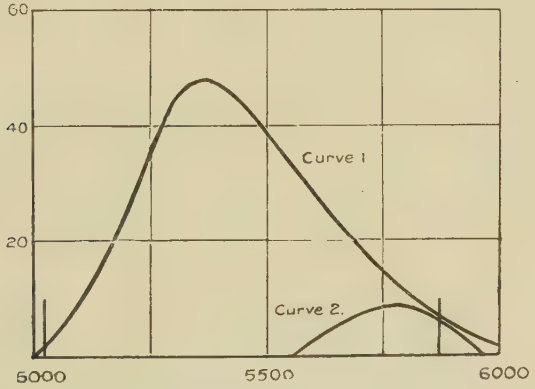
† An account of other investigations on the positive column in gas discharges has been given recently by S. P. MacCallum and F. Llewellyn-Jones, Phil. Mag. xii. p. 384 (Aug. 1931).

‡ The band spectrum is conspicuous near the electrodes under the conditions described in the paper by J. S. Townsend and S. P. MacCallum, Phil. Mag. xii. p. 1168 (Dec. 1931).

intensity of the continuous spectrum. There is thus a large range of wave-lengths where the light is continuous in the helium spectrum, and by using suitable filters it is possible to separate the photoelectric effect of the green continuous spectrum from that of the yellow and green lines. The changes in the intensities of the continuous spectrum and the line spectrum due to changes in pressure may thus be found.

5. For this purpose two green filters and two yellow filters, made by Messrs. Ilford, were used. The transmission curves of the filters are shown by the curves in fig. 1, where the ordinates represent the ratio of the transmitted to the incident light and the abcissæ the wave-lengths. The two

Fig. 1.



broad lines in the figure represent the positions of the principal helium lines in this part of the spectrum.

The two green filters were practically identical, and the transmission for each is given by curve 1. Each of these filters transmits about 45 per cent. of green light between the wave-lengths $\lambda 5280$ and $\lambda 5480$, and about 5 per cent. of the light of the yellow line $\lambda 5876$. The proportion of the light in the green helium line $\lambda 5016$ that is transmitted is very small. The effects of the green line may be completely eliminated from the effect of the continuous green light and the yellow line by using two green filters.

These filters also transmit a small proportion of red light of wave-lengths exceeding $\lambda 7000$ (beyond the range of wave-lengths in the diagram fig. 1) : but the lines in this part of the spectrum of helium are very faint. The photo-

electric effect due to them is easily found, and the necessary small correction may be made when using one of the green filters, but with two green filters the effect of the red light from helium discharges is negligible.

The sensitivity of the photoelectric cell used in these experiments was nearly the same for yellow and green light. The photoelectric effect of the light from a discharge in helium that traverses one of the green filters was found to be principally due to the continuous green light and the light of the yellow line. With discharges at pressures of about 3 mm. the effect of the yellow line is much greater than that of the continuous light, but at pressures of about 20 mm. the effect of the continuous light is much greater than that of the yellow line.

6. The two yellow filters that were used were also practically identical. The transparency of each is given by curve 2. It will be observed that each of these filters transmits about 5 per cent. of the light in the yellow line and about 8 per cent. of the light between the wave-lengths λ 5700 and λ 5800. The effect of the light between these wave-lengths may be neglected in comparison with that of the yellow line when the pressure is less than 12 mm., and the photoelectric effect obtained with light from helium discharges that has passed through the yellow filter may be attributed to the light in the yellow line λ 5876.

7. The continuous spectrum may be obtained either from the uniform positive column of a direct-current discharge or from the uniform luminous column of a high-frequency discharge. The mean energy of agitation is the same in the two types of discharge and is constant over the whole length of the uniform column. The high-frequency discharges, however, are much the simplest as there are no internal electrodes in the tubes. The discharges, which are maintained with external electrodes, are perfectly continuous, and there is a uniform glow in most of the space between the electrodes, without any dark space corresponding to the Faraday dark space in direct-current discharges. The investigations of the spectrum were therefore made with electrodeless quartz tubes containing pure helium at different pressures.

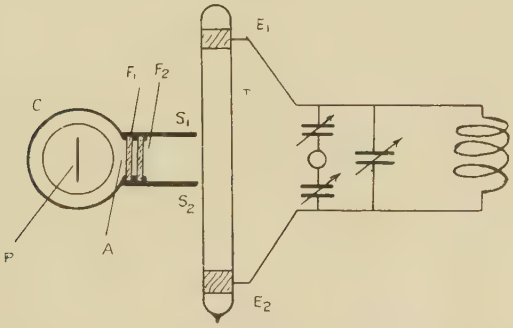
The discharge-tube was connected to an oscillatory circuit as shown in fig. 2. The external electrodes, E_1 and E_2, of the tube T, containing the helium, were connected to terminals of the inductance of an oscillatory circuit. The circuit included two condensers connected in series through a thermo-junction,

for the purpose of measuring the potential difference between the electrodes E_1 and E_2. The circuit was loosely coupled to a continuous wave generator maintained in oscillation by three-electrode valves, the wave-length of the oscillations being 100 metres.

The method of measuring the electric force and adjusting the current in the discharge-tube with this arrangement has already been described in the accounts of previous experiments *.

The current in the tube was measured by the induced current in a circuit consisting of two small loops, l_1 and l_2, connected as shown in fig. 3. The loops were in the same plane, and the axis of the tube, AA, was midway between the loops and in the plane of the loops. With this arrangement

Fig. 2.



there was a resultant electromotive force in the loops due to a current in the tube ; but the force in the loops due to currents in other parts of the oscillatory circuit was very small. The current in the loops was therefore proportional to the current in the discharge-tube. The loops were connected to two mica condensers of large capacity, so as to form a small oscillatory circuit in resonance with the generator, the current in the circuit being measured by the thermo-junction J. The current induced in the loops by a known current in a wire midway between the loops was also determined, and the current in the discharge-tube was thus found.

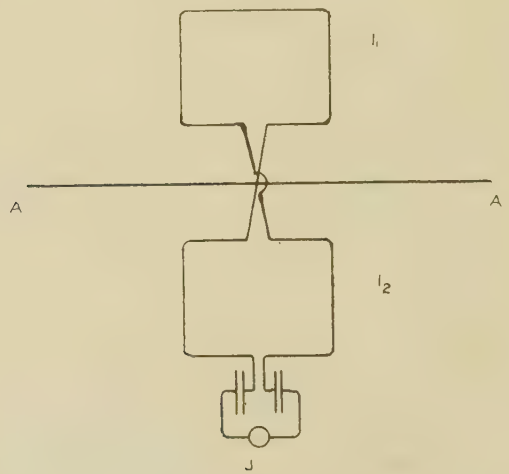
The photoelectric cell (P, fig. 2) was contained in a metal case C, with an aperture, A, to admit the light from the

* J. S. Townsend and R. H. Donaldson, Phil. Mag. v. p. 178 (Jan. 1928) : R. L. Hayman, Phil. Mag. vii. p. 586 (March 1929).

discharge-tube. A rectangular screen, S_1 S_2, of ebonite was fitted to the aperture, so that the light from the uniform column of the discharge fell on the photoelectric cell, but the light from parts of the discharge near the electrodes was excluded. There were two sets of grooves in the screen for filters, so that either one filter or two filters could be placed across the path of the light that fell on the photoelectric cell. In the figure two filters, F_1 and F_2, are shown in position.

The photoelectric currents were measured by an electrostatic balance connected to an electrometer. The condenser of the system was adjustable, so that currents of different

Fig. 3.



orders of magnitude, from 10^{-13} to 10^{-12} amp., could be measured to the same degree of accuracy.

8. In order to compare the intensity of the continuous green spectrum with the intensity of the yellow line it is necessary to find exactly the transparencies of the yellow and green filters for the yellow line. For this purpose a tube, 4 cm. in diameter, containing helium at 1 mm. pressure was used. At that pressure (and at lower pressures) the intensity of the line spectrum is very large compared with that of the continuous light or of the bands. The yellow filter was quite opaque to all the lines except the yellow line, so that a monochromatic yellow beam is obtained by

passing the light from the discharge through one of the yellow filters. The transparency of the second yellow filter was determined by measuring the photoelectric current Y_1, due to light that had passed through one filter and the current $(Y_1 Y_2)$ due to light that had passed through the two filters. The photoelectric current $(Y_1 G_1)$, due to light that had passed through one yellow and one green filter, was also measured.

The transparency $(Y_1Y_2)/Y_1$ of the yellow filter for the yellow line was thus found to be 0·049, and that of the green filter $(Y_1 G_1)/Y_1$ was 0·047. Thus the yellow filter lets through a little more yellow light than the green filter. If the photoelectric current due to a beam of yellow light after passing through one yellow filter be 100, the photoelectric current would be 96 if the beam had passed through one of the green filters instead of the yellow filter, and the

TABLE I.

Helium, pressure 12·5 mm.

Y_1.	G_1.	$G_1 G_2$.	C_1.	$C_1 C_2$.
14·3	34	8·4	20	7·7
26·2	62	15·5	37	14·2
100	237	59	141	54·5

current would be 4·5 if the beam had passed through two green filters.

9. With tubes containing helium at higher pressures the effect of the continuous spectrum becomes prominent, and the photoelectric effect of the light from a discharge that traverses one of the green filters is much greater than the effect of the light that traverses a yellow filter.

This is seen by the figures in Table I., which gives the results of the experiments with a tube 4 cm. in diameter containing helium at 12·5 mm. pressure. The first column (Y_1) gives the photoelectric current due to light that had passed through one yellow filter; the second column (G_1) gives the current due to light that had passed through one green filter; and the third column $(G_1 G_2)$ gives the current due to light that had passed through two green filters. The first line of figures in the table represent the photoelectric effects obtained with the current of $1·4 \times 10^{-3}$ amp.

in the tube, and the second line of figures are the effects with a current of $2{\cdot}6\times10^{-3}$ amp. In other experiments with high-frequency discharges it was also found that with small currents the photoelectric effect of the yellow line is proportional to the current in the discharge when the pressure of the gas is constant. The figures in Table I. show that the intensity of the continuous spectrum is also proportional to the current when the current is small.

The experiments on the transparencies of the filters for the yellow light show that the effect of the yellow line in the photoelectric current G_1 is $96Y_1/100$. The remainder $(G_1-96Y_1/100)$ is principally due to the continuous spectrum, but it also includes a small effect due to the strong green line and the red line. Thus, if C_1 be the photoelectric current of the continuous light that traverses one green filter, the values of C_1 are approximately $G_1-96Y_1/100$. Similarly

Table II.

p.	Y_1.	G_1.	$(G_1\,G_2)$.	C_1.	$(C_1\,C_2)$.
27	100	480	180	384	175
23	100	410	132	314	127
12·5	100	237	59	141	54·5
3	100	141	19·2	45	14·7

with the photoelectric current G_1G_2. The effect of the yellow line in this current is $4{\cdot}5Y_1/100$, and as the two filters are much more selective than one, the proportion of the green and red lines that traverses them is negligible, so that the effect of the continuous light that traverses the two green filters $(C_1\,C_2)$ is accurately proportional to the difference $[(G_1\,G_2)-4{\cdot}5Y_1/100]$.

The numbers in the tables show that Y_1, G_1, and $(G_1\,G_2)$ are proportional to the numbers 100, 237, and 59, so that the values of C_1 and $(C_1\,C_2)$ are proportional to the numbers 141 and 54·5.

10. Similar experiments were made with tubes about 4 cm. in diameter containing helium at the pressures 3, 23, and 27 mm. The results are given in Table II., where the photoelectric current Y_1 due to the light that traverses one yellow filter is taken in each case to be 100. The pressure p of the gas is given in the first column.

The rate of increase of the intensity of the continuous spectrum in comparison with the intensity of the yellow line as the pressure increases is shown approximately by the figures in the column C_1, and more accurately by the figures in the column $(C_1\ C_2)$, as the effects of the green and red lines are more completely eliminated from the currents $(C_1\ C_2)$. The figures show that the continuous spectrum increases in comparison with the yellow line by the factor 12 as the pressure changes from 3 to 27 mm.

11. The currents in the helium at the different pressures were also measured, and it was found that with the same current in the discharge-tubes the photoelectric effect of the yellow line diminished and that of the continuous spectrum increased as the pressure was increased. These results are shown in Table III., where y represents the intensity of the

TABLE III.

p.	y.	c.	Z.	E.	$W \times 10^{-5}$.
27	100	100	37·5	2·78	3·25
23	140	100	32·6	2·84	3·3
12·5	320	98	19	3·04	3·5
3	850	71	6·4	4·26	4·0

yellow line and c the intensity of the continuous spectrum, the intensity being taken as 100 at the pressure of 27 mm. in each case.

The electric force Z in the direction of the axis of the tube, as found experimentally, is given in the fourth column of the table. The numbers represent the root-mean-square values of the force in volts per centimetre.

The mean energy of agitation E of the electrons is obtained from the ratio Z/p. It is the same as the mean energy E in a direct-current discharge maintained by a force Z. In the wide tubes containing gas at several millimetres pressure used in these experiments the collisions of the electrons with the sides of the tube have very little effect on the mean energy of agitation compared with the effect of the collisions with atoms of the gas. Thus the mean energy of agitation of the electrons in the discharges is practically the same as the mean energy obtained with very small currents moving under a direct force Z in a wide

space where there are no collisions with the boundary. The principal difference is due to the radial force R, but with discharges in tubes of large diameter containing gas at high pressures it may be seen that the effect of this force on the mean energy of the electrons is negligible. It has been shown * that the potential difference $\int_0^a R\,dr$ between a point near the surface and a point on the axis of the tube is $(2E \log A/A')/3$, where A/A' is the ratio of the number of electrons per cubic centimetre at the axis to the number near the surface. This ratio has not yet been found accurately, but if A be assumed to be $2A'$ the potential $\int_0^a R\,dr$ is ·46E.

In order to estimate the effect of the force R the value of E may be taken to be the same as the mean energy of the electrons in the experiments † with small currents, where the velocities were determined in terms of the ratio Z/p. For the ranges of forces and pressures in Table III. the energy E in volts is approximately $2Z/p$ and the mean value of the force R is $\cdot 92 \times Z/ap$. Thus R diminishes in comparison with Z as the pressure increases. In these experiments the greatest effect of the force R is at the pressure of 3 mm., where $R = \cdot 16 \times Z$. The resultant of this force and the force Z is $1\cdot 012 \times Z$, so that it may be assumed that the mean energy of agitation of the electrons is given to a sufficient degree of accuracy by the formula $E = 2Z/p$.

The velocity of the electrons in the direction of the electric force depends on the energy of agitation. If W be the mean velocity in the steady motion when the electrons attain the mean energy of agitation E under the action of a direct force Z, the velocity in the high-frequency currents is $W\sqrt{2}\sin pt$, where $p/2\pi$ is the frequency.

The values of W in the discharges at the different pressures are given in the last column of Table III.

12. Since the intensity of the continuous radiation and the intensity of the yellow line are proportional to the current in the gas, both types of radiation may be attributed to the collisions of electrons with atoms of the gas, and the mean energies of the electrons that excite the different types of radiation may be estimated by the method that has already

* Phil. Mag. (7) xiii. p. 745 (April 1932).

† J. S. Townsend and V. A. Bailey, Phil. Mag. xlvi. p. 657 (Oct. 1923).

been explained *. The exact values of these energies cannot be found, as there is no accurate formula for the distribution of the energies of the electrons about the mean.

In this case it is of more interest to compare the mean energy E_c of the electrons in the collisions which excite the continuous radiation with the mean energy E_y in the collisions which excite the yellow line, as the difference between these two quantities is so large.

The general method of analyzing the results of the experiments is as follows:—

Let N be the total number of electrons in unit section of the tube, E the mean energy of agitation, and N_x the number with energies greater than E_x. The latter number depends on the ratio E_x/E, and may be expressed in the form $N_x = N\phi\ E_x/E)$. The number of collisions of electrons per second with atoms of the gas in which the energies of the electrons is greater than E_x is proportional to $N_x \times p$.

If W be the velocity of the electrons in the direction of the electric force the product $N \times W$ is constant when the same current flows in tubes containing gas at different pressures, so that N is inversely proportional to W. Also, since the intensity c of the radiation from unit section of the tube is proportional to the product $N_c \times p$, the following expression is obtained for c:—

$$c = B(p/W)\phi(E_c/E),$$

where B is a constant.

If c_1 be the intensity of the radiation, W_1 the velocity in the direction of the force Z, E_1 the mean energy of agitation of all the electrons with a given current in the gas at pressure p_1, and c_2, W_2, and E_2 the corresponding quantities with the same current in the gas at pressure p_2, the following equation is obtained for the energy E_c:—

$$\frac{c_1}{c_2} = \frac{p_1 W_2}{p_2 W_1} \times \frac{\phi(E_c/E_1)}{\phi(E_c/E_2)}.$$

Taking the results of the experiments at 12·5 mm. and 3 mm. given in Table III., the equation for the mean energy E_c of the electrons that excite the continuous radiation becomes

$$\phi(E_c/4{\cdot}26) = 3{\cdot}45 \times \phi(E_c/3{\cdot}04).$$

Similarly the equation for the mean energy E_y of the electrons that excite the yellow line becomes

$$\phi(E_y/4{\cdot}26) = 12{\cdot}6 \times \phi(E_y/3{\cdot}04).$$

* J. S. Townsend and F. Llewellyn-Jones, and P. Johnson, *loc. cit.*

Thus the rate of change of the ratio N_c/N with the mean energy E is much less than the rate of change of N_y/N, for the range of values of E between 4·26 and 3·04 volts, which shows that the ratio E_y/E is much greater than E_c/E. The numerical results obtained from equations of this type depend on the formula taken for the distribution of the energies of the electrons.

Examples of the results obtained from different formulæ for the distribution have been given in previous papers *.

13. It must be observed that the distribution is not correctly represented by the Maxwellian distribution when the electrons are moving in an electric field and the mean energy of agitation of the electrons is about 100 times the mean energy of agitation of the molecules of the gas. In this case the inequalities in the energies are principally due to the different rates of diffusion of electrons in the direction of the electric force, and the causes which give rise to the Maxwellian distribution are of less importance than in cases of gases in thermal equilibrium.

The formula for the distribution due to diffusion, in which the other causes of inequalities are not taken into account, gives a rate of change of the function ϕ which is too large, so the values of E_x obtained from this formula are too small.

The formula

$$\phi(E_x/E) = A \sqrt{E_x/E}\, \epsilon^{-3E_x/2E},$$

which is approximately Maxwell's formula for energies which are large compared with the mean energy, gives a more gradual change of the function ϕ. The values of E_c and E_y, obtained by taking this expression for ϕ, are

$$E_c = 10 \text{ volts} \quad \text{and} \quad E_y = 19 \text{ volts}.$$

These values are more in accordance with the results of other experiments, provided it be assumed that large losses of energy occur only in a very small proportion of the total number of collisions in which the energy of the electrons is about 10 volts.

Thus in the experiments on the motion of electrons, where the mean energy of agitation was determined, it was found that there was no appreciable effect of large losses of energy when the ratio Z/p was less than 2, but there was a considerable increase in the mean loss of energy in collisions when the ratio Z/p was greater than 2—that is, when the mean energy of agitation is greater than 4 volts.

* Phil. Mag. ix. p. 1145 (1930); xi. p 679 and p. 1112 (1931); xii. p. 815 (1931); and xiii. p. 487 (1932).

An increase in the mean loss of energy would become noticeable when the mean energy is 4 or 5 volts if some of the electrons which acquired energies of about 10 volts lost a large proportion of their energies in collisions; but when the mean energy is 3 or 4 volts the number of electrons with energies of about 10 volts would be much smaller, and the amount of energy they lose would have very little effect on the mean loss of energy.

XLVII. *A Theorem on States in Kinetic Equilibrium.*

By V. V. Narliker, *Isaac Newton Student, Cambridge* *.

CONSIDER a conservative holonomic dynamical system with n cyclic coordinates $q_1, q_2, \ldots q_n$ and any number of "palpable" coordinates $\theta_1, \theta_2, \theta_3, \ldots$ etc. Let $V = V(\theta)$ be the potential energy. If $p_1, p_2, \ldots p_n$ are the momenta corresponding to $q_1, q_2, \ldots q_n$ respectively we have relations of the type

$$p_1 = \sum_{r=1}^{n} a_{r_1}\dot{q}_r + \kappa_{11}\dot{\theta}_1 + \ldots,$$

$$p_2 = \sum_{r=1}^{n} a_{r_2}\dot{q}_r + \kappa_{12}\dot{\theta}_1 + \ldots, \text{ etc.}$$

The coefficients of the velocities in the above relations are in general functions of θ's. We have also $a_{rs} = a_{sr}$, etc. Let $\theta_1 = \alpha_1$, $\theta_2 = \alpha_2, \ldots$ etc. give a position of kinetic equilibrium in which $\dot{q}_1 = \omega_1$, $\dot{q}_2 = \omega_2, \ldots$ etc., $p_1 = \beta_1$, $p_2 = \beta_2, \ldots$ and $p_n = \beta_n$. Also let $a_{rs}(\theta)$ reduce to b_{rs}, so that

$$\beta_1 = \sum_{r=1}^{n} b_{r_1}\omega_r, \quad \beta_2 = \sum_{r=1}^{n} b_{r_2}\omega_r, \ \ldots \text{ etc.}$$

If $\{b^{rs}\}$ is the matrix reciprocal to $\{b_{rs}\}$,

$$\omega_1 = \sum_{r=1}^{n} b^{r_1}\beta_r, \quad \omega_2 = \sum_{r=1}^{n} b^{r_2}\beta_r, \ \ldots \text{ etc.}$$

If the velocities $\omega_1, \omega_2, \ldots$ etc. are assigned, the kinetic energy is of the form

$$T = \Sigma\Sigma\tfrac{1}{2}a_{rs}\omega_r\omega_s + \tfrac{1}{2}K_1\dot{\theta}_1^2 + L_1\dot{\theta}_1\omega_1 + \ldots,$$

so that in the position of equilibrium

$$T = T_0 = \Sigma\Sigma\tfrac{1}{2}b_{rs}\omega_r\omega_s.$$

* Communicated by the Author.

If, on the other hand, the cyclic momenta $\beta_1, \beta_2, \ldots$ etc. are assigned, the kinetic energy is of the form

$$T' = \Sigma\Sigma\tfrac{1}{2}a^{rs}\beta_r\beta_s + \tfrac{1}{2}K_1\dot{\theta}_1^2 + L_1'\dot{\theta}_1\beta_1 + \ldots,$$

so that in the position of equilibrium

$$T' = T_0' = \Sigma\Sigma\tfrac{1}{2}b^{rs}\beta_r\beta_s.$$

We will assume now for the dynamical states under consideration that the parts of T and T′ which are due to the "palpable" coordinates are only quadratic in $\dot{\theta}$'s, so that $L_1 = L_1' = 0$, etc. Then the theorem states that if $V - T$ (*i. e.*, the real "plus" fictitious energy when the cyclic velocities are assigned) is a minimum $V + T'$ (*i. e.*, the real "plus" fictitious energy when the cyclic momenta are assigned) is also a minimum; but if the latter is a minimum it does not necessarily follow that the former is a minimum. The appreciation of this last point is due to Sir Horace Lamb, who suggested to me the symmetrical top as an illustration.

To prove the theorem it is sufficient to show that

$$V - T > V_0 - T_0$$

is not compatible with

$$V + T' < V_0 + T_0'.$$

If the two inequalities hold good simultaneously,

$$T_0' - T' > V - V_0 > T - T_0,$$

or

$$T_0' + T_0 - T' - T > 0.$$

From this follows the inequality

$$\Sigma_s\Sigma_r(b^{rs}\beta_r\beta_s + b_{rs}\omega_r\omega_s - a_{rs}\omega_r\omega_s - a^{rs}\beta_r\beta_s) > 0,$$

i. e.,

$$\Sigma_s\Sigma_r(2b_{rs}\omega_r\omega_s - a_{rs}\omega_r\omega_s - \Sigma_{s'}\Sigma_{r'}a^{r's'}b_{rr'}b_{ss'}\omega_r\omega_s) > 0.$$

Therefore

$$-\Sigma_{s'}\Sigma_{r'}\Sigma_s\Sigma_r\{(a_{rr'} - b_{rr'})(a_{ss'} - b_{ss'})a^{r's'}\omega_r\omega_s\} > 0.$$

But obviously this is absurd, since

$$\Sigma_{s'}\Sigma_{r'}a^{r's'}x_{r'}x_{s'}$$

is a positive definite form. This means that the two inequalities from which we started are not compatible with one another. Thus the theorem is proved.

The preceding analysis also suggests that when $V + T'$ is a maximum $V - T$ is necessarily a maximum.

The method of argument used here is due to Poincaré, who uses it in his 'Figures d'équilibre,' p. 37 (1902) for discussing the criteria of secular stability for a rotating liquid mass.

XLVIII. *A World Criterion.*

By V. V. Narliker, *Isaac Newton Student, Cambridge* *.

Abstract.

A criterion is given to distinguish a mass-point from all other mathematically possible singularities. The same criterion shows that during the passage from the Einstein state to another, where all the cosmical matter has broken up into discrete condensations, λ increases from $\frac{1}{R_0^2}$ to $\frac{3}{2}\frac{1}{R_0^2}$. That we observe most of the cosmical matter as discrete condensations is therefore taken as evidence for a tendency in λ to increase in the Einstein state. This is shown mathematically to be equivalent to an initial tendency in the universe to expand. It is thus possible that the genesis of the stellar system was the cause of the expansion of the universe; so that the stellar system of our observation may be evidence enough about the expansion of the universe if the interpretation of the red-shift of the spiral nebulæ is controversial.

LONG before the theory of non-static universe was proposed, Einstein's cosmology was abandoned on grounds that a condensation could not be formed in it without introducing non-transformable "impossibilities" in the world structure. In any static universe, spherical and homogeneous, the field outside an isotropic mass-singularity is subject to three relativity equations connecting the pressure p, the density ρ, and α and β, where

$$ds^2 = -e^{\alpha}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\theta^2) + e^{\beta}dt^2 \quad r > a$$

and $\quad T_1^1 = T_2^2 = T_3^3 = -p, \quad T_4^4 = \rho, \quad T_{\mu}^{\nu} = 0, \quad \mu \neq \nu.$

a is the radius of the mass. As the relativity equations fall short by one for a complete solution, one may anticipate that all singularities subject to these equations cannot be identified with mass-points whose properties are familiar. An assumption connecting p, ρ, α and β is necessary.

De Sitter [1], who (it seems) was the first to look into this question, assumed ρ = constant. The apparent difficulties which creep into his solution are at the polar of the mass-point; they are real and cannot be transformed away. Also, in view of the interaction between a mass-point and space

* Communicated by the Author.

the assumption seems artificial that the cosmic density should remain unaffected by the formation of a condensation (while the pressure becomes a function of r). McVittie and McCrea[2], while considering the effect of a single condensation in the Einstein world on the total space-volume, make an assumption about β. This assumption is also very unsatisfactory, particularly because one doubts as to whether the singularity considered by them is a mass-point at all. This is a vital point in the determination of the effect of condensations on the volume of the three-dimensional space, since mathematical singularities of different types may lead to contradictory results.

A criterion is therefore necessary to distinguish a mass-point from the other mathematically possible, singularities that inevitably arise in the solution of Einstein's equations. This criterion, to be furnished by the external field, must be of a tensorial form ; for it is only then that any peculiarities of the external world which are not desirable can be transformed away, if at all. A suitable criterion is furnished by the world curvature $G = kT + 4\lambda$. We see at once that a mass-point in empty space does not change the world curvature. Extending this idea to the world full of matter and motion the criterion for a mass-point may be taken so that the world-curvature is not disturbed by it. This is the extra condition necessary to supplement the field equations to determine p, ρ, α and β. In the isotropic field of a variable mass the outward flow of matter which is otherwise arbitrary becomes determinate when this criterion is applied. A solution in series can be easily obtained.

A justification for the proposed criterion is that the latter can be used to remove some of the difficulties in connexion with the theory of the expansion of the universe. Eddington[3] has shown that the Einstein world is unstable. It is natural to assume that the manner in which the instability sets in should account both for the origin of the stellar system and the expansion of the universe. This suggests that the origin of the stellar system, in other words, the formation of condensations in the Einstein world may account for the expansion of the universe. No successful attempt has yet been made to prove this mathematically.

It seems possible to solve the problem by supplementing with the criterion, the Lemaître equations of the universe, which, as de Sitter[4] has remarked, are not sufficient for a complete theory of the expanding universe. The reason for this procedure is that the formation of condensations is a process which leaves undisturbed the world curvature

outside the condensations. When the Einstein universe becomes unstable we have the Lemaître [5] equations

$$\lambda + k\rho = \frac{3}{R^2} + \frac{3R'^2}{R^2},$$

$$\lambda - kp = \frac{1}{R^2} + \frac{R'^2}{R^2} + \frac{2R''}{R},$$

and the energy equation

$$\frac{d}{dt}(\rho R^3) = -p\frac{dR^3}{dt}.$$

The criterion gives

$$G \equiv 6\left(\frac{1}{R^2} + \frac{R'^2}{R^2} + \frac{R''}{R}\right) = \frac{6}{R_0^2},$$

where R_0 is the radius of the Einstein universe. Solving the last equation we get

$$R^2 = R_0^2 + A\left(e^{\frac{\sqrt{2}}{R_0}t} - e^{-\frac{\sqrt{2}}{R_0}t}\right).$$

Hence

$$\lambda - kp - \frac{1}{R_0^2} = \frac{R''}{R} = \frac{2A}{R^2R_0^2}\sinh\frac{\sqrt{2}}{R_0}t - \frac{2A^2}{R^2R_0^2}\cosh^2\frac{\sqrt{2}}{R_0}t.$$

A is to be treated as a small parameter, since the Lemaître equations do not hold exactly when condensations are being formed in the cosmical matter. The universe will begin to expand or contract according as A is greater or less than zero. Since

$$G \equiv kT + 4\lambda = \frac{6}{R_0^2}$$

as $T \to 0$, which happens when the whole cosmical matter splits up into condensations, $\lambda \to \frac{3}{2}\frac{1}{R_0^2}$. This means that λ increases from $\frac{1}{R_0^2}$ to $\frac{3}{2}\frac{1}{R_0^2}$ during the period of genesis of the stellar system. We therefore assume that initially

$$\delta\lambda = \lambda - kp - \frac{1}{R_0^2}$$

is positive, which gives a positive value of A. If $\delta\lambda$ is positive, δT is negative, which shows that the effect of the formation of condensations is to diminish the cosmical density T. Thus the genesis of the stellar system may possibly account for the origin of the expansion of the universe.

We assume that the cosmical constant λ is in the initial stages a function of time, the behaviour of which is controlled by physical factors which are outside the present theory[6]. This leaves open the possibility of reconciling the two time-scales[7], as considerations about radioactive elements have been mainly responsible for such a long time-scale for the stars. In our considerations of the instability of the universe we had to stick to a state in the neighbourhood of the Einstein state, as a universe in which several condensations have already been formed is probably too irregular to be within the scope of Lemaître's equations. But the departure from these equations will proceed according to some physical criterion which has been, perhaps justly, ignored in the considerations here.

References.

(1) Silberstein, 'The Size of the Universe,' pp. 73–84 (1930).
(2) McVittie and McCrea, M. N. R. A. S. xci. pp. 128, 274.
(3) Eddington, M. N. R. A. S. xc. p. 668.
(4) De Sitter, Proc. N. A. S. xvi. p. 488 (1930).
(5) Lemaître, M. N. R. A. S. xci. p. 485.
(6) The energy equation of Lemaître is identically satisfied when λ is a constant; it must be modified when λ is variable.
(7) Suppl. to 'Nature,' Oct. 24, 1931: discussion on the Evolution of the Universe.

XLIX. *The Stresses in a Wire Wheel under Side Loads on the Rim.* *By* Prof. A. J. Sutton Pippard, *M.B.E., D.Sc., and* W. E. Francis, *M.Sc.**

IN previous papers† solutions have been given for the stresses in a wire wheel when loads are applied to the rim, in the plane of the wheel. The same methods of analysis are here applied to the case when the rim load is normal to that plane.

The wheel is supposed to have a large number of spokes which may be either radial or non-radial. Half of these spokes are inclined inwards and a half outwards as shown in fig. 1 and as before, we assume them to be replaced for

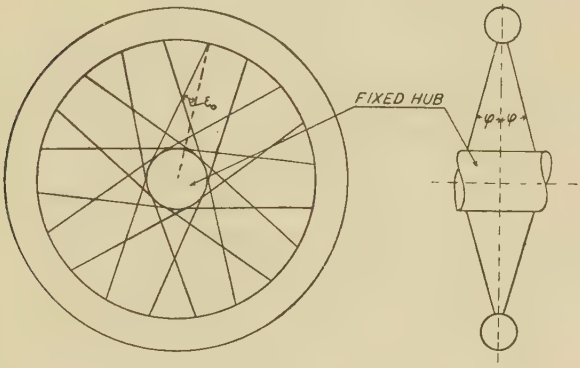
* Communicated by the Authors.

† "The Stresses in a Radially Spoked Wire Wheel under Loads Applied to the Rim." A. J. S. Pippard and W. E. Francis, Phil. Mag. xi. Feb. 1931, pp. 233-285; "The Stresses in a Wire Wheel with Non-radial Spokes under Loads Applied to the Rim." A. J. S. Pippard and Miss M. J. White, Phil. Mag. xiv. p. 209 (1932).

the purpose of analysis by two equivalent disks having peculiar elastic properties. The solutions obtained thus apply to the case of an infinite number of spokes but constitute good approximations for actual wheels.

Let ϵ_0 be the angle which the spokes make with the radius in the plane of the wheel and ϕ the angle of inclination to the plane of the wheel, the forward and backward inclination being taken to be the same.

Fig. 1.



Let R be the radius to the centroid of the rim,

R_0 the radius of the tangent circle of the hub,

I′, the moment of inertia of the rim about its axis of bending,

a, the cross sectional area of the rim,

E_1, the modulus of elasticity of the rim material,

E_2, the modulus of elasticity of the spoke material,

CJ, the torsional rigidity of the rim, and

t, the equal tension and compression per unit length of rim applied by the disks at any point.

From the assumption of symmetry about the mid plane of the wheel it follows that there will be no radial or tangential movement of any point on the rim, *i. e.*, *u* and *v* of the previous analysis are both zero.

At any point on the rim at an angle ψ from X0 in fig. 2 let the movement normal to the plane of the wheel be *w* taken positive in the direction of the load.

Also, let N be the bending moment perpendicular to this plane,

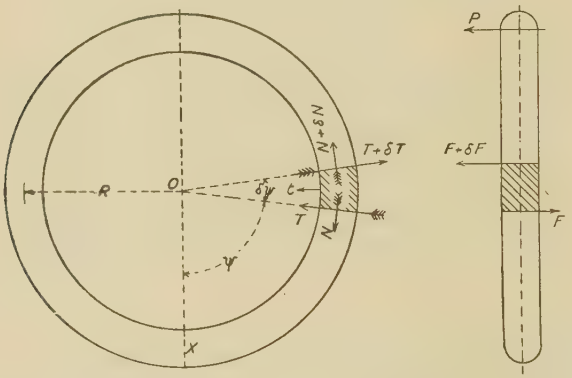
F the shearing force perpendicular to this plane,

and T the torque on the rim.

Further, let χ be the cosine of the angle between the original direction of a radial axis at any point on the rim and the final direction of the normal axis through this point.

The equations of equilibrium of a small element of the rim as shown in fig. 2 are *

Fig. 2.



$$\frac{\partial \mathrm{F}}{\partial \psi} = 2\mathrm{R}t \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\partial \mathrm{N}}{\partial \psi} + \mathrm{T} - \mathrm{FR} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{\partial \mathrm{T}}{\partial \psi} = \mathrm{N}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\mathrm{N} = \frac{\mathrm{EI}'}{\mathrm{R}^2}\left(\chi \mathrm{R} - \frac{\partial^2 w}{\partial \psi^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\mathrm{T} = \frac{\mathrm{CJ}}{\mathrm{R}^2}\left(\mathrm{R}\frac{\partial \chi}{\partial \psi} + \frac{\partial w}{\partial \psi}\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

also

$$t = \frac{\Delta \mathrm{E}_2 w \tan \phi}{\mathrm{R}(1 + \cos^2 \epsilon_0 \tan^2 \phi)} \quad . \quad . \quad . \quad . \quad (6)$$

* 'The Mathematical Theory of Elasticity,' A. E. H. Love, 3rd edition, p. 457.

Writing $$K_2 = \frac{2R^3 \Delta E_2 \tan\phi \sin\phi}{EI(1+\cos^2\epsilon_0 \tan^2\phi)}$$

and $$n = \frac{EI'}{CJ},$$

we obtain from the above equations,

$$\frac{\partial^6\chi}{\partial\psi^6} + \frac{2\partial^4\chi}{\partial\psi^4} + \frac{\partial^2\chi}{\partial\psi^2}(1+K_2) - n\chi K_2 = 0 \quad . \quad . \quad (7)$$

and $$\frac{\partial^2 w}{\partial\psi^2} = \frac{R}{n+1}\left(n\chi - \frac{\partial^2\chi}{\partial\psi^2}\right) \quad . \quad . \quad . \quad . \quad (8)$$

The complete solution is

$$\chi = G\cosh\gamma\psi + H\cos\alpha\psi\cosh\beta\psi + U\sin\alpha\psi\sinh\beta\psi + G'\sinh\gamma\psi + H'\cos\alpha\psi\sinh\beta\psi + U'\sin\alpha\psi\cosh\beta\psi,$$

where $$\alpha = \sqrt{\frac{\left(\frac{X^2}{4} + \frac{2X}{3} + \frac{4}{9} + \frac{3Y^2}{4}\right)^{\frac{1}{2}} + \left(\frac{X}{2} + \frac{2}{3}\right)}{2}}$$

$$\beta = \sqrt{\frac{\left(\frac{X^2}{4} + \frac{2X}{3} + \frac{4}{9} + \frac{3Y^2}{4}\right)^{\frac{1}{2}} - \left(\frac{X}{2} + \frac{2}{3}\right)}{2}}$$

$$\gamma = (X - \tfrac{2}{3})^{\frac{1}{2}},$$

$$X = (M+Q)^{\frac{1}{3}} + (M-Q)^{\frac{1}{3}},$$

$$Y = (M+Q)^{\frac{1}{3}} - (M-Q)^{\frac{1}{3}},$$

$$M = \left\{\frac{1}{27} + \frac{K^2(3n+2)}{6}\right\},$$

$$Q = \sqrt{\frac{K_2}{27}\left[K_2^2 + 2K_2 + 1 + n\left(\frac{27nK_2}{4} + 9K_2 + 1\right)\right]}.$$

Since χ is the same for equal positive and negative values of ψ, the constants G', H' and U' are zero, so,

$$\chi = G\cosh\gamma\psi + H\cos\alpha\psi\cosh\beta\psi + U\sin\alpha\psi\sinh\beta\psi \, . \quad (9)$$

$$w = \frac{R}{n+1}\left[\frac{n-\gamma^2}{\gamma^2} G\cosh\gamma\psi - \frac{\cos\alpha\psi\cosh\beta\psi}{(\alpha^2+\beta^2)^2}\left\{H\left(\sqrt{\alpha^2+\beta^2}^{\,2} + n.\alpha^2 - \beta^2\right) + 2\alpha\beta n U\right\} - \frac{\sin\alpha\psi\sinh\beta\psi}{(\alpha^2+\beta^2)^2}\left\{U\left(\sqrt{\alpha^2+\beta^2}^{\,2} + n.\alpha^2 - \beta^2\right) - 2\alpha\beta n H\right\}\right] + E\psi + F \, . \quad . \quad . \quad (10)$$

The constants are determined from the following conditions.

(*a*) By equation of forces normal to the rim

$$4R \sin \phi \int_0^{\pi} t d\psi = P,$$

(*b*) By equation of moments about a diameter of the wheel

$$4R \sin \phi \int_0^{\pi} t \cos \psi \, d\psi = -P.$$

(*c*) When $\psi = 0$, $\dfrac{\partial w}{\partial \psi} = 0.$

(*d*) When $\psi = \pi$, $\dfrac{\partial w}{\partial \psi} = 0.$

(*e*) When $\psi = \pi$, $\dfrac{\partial \chi}{\partial \psi} = 0.$

These give :

$$G = Z_1 \left(\frac{2\alpha\beta\gamma}{\sinh \gamma\pi} \right) = Z_1 G_1,$$

$$H = -Z_1 \left\{ \frac{\begin{array}{l}\alpha(\alpha^2+\beta^2+\gamma^2) \cos \alpha\pi \sinh \beta\pi \\ \quad +\beta(\alpha^2+\beta^2-\gamma^2) \sin \alpha\pi \cosh \beta\pi\end{array}}{\cosh^2 \beta\pi - \cos^2 \alpha\pi} \right\} = -Z_1 H_1$$

$$U = -Z_1 \left\{ \frac{\begin{array}{l}\alpha(\alpha^2+\beta^2+\gamma^2) \sin \alpha\pi \cosh \beta\pi \\ \quad -\beta(\alpha^2+\beta^2-\gamma^2) \cos \alpha\pi \sinh \beta\pi\end{array}}{\cosh^2 \beta\pi - \cos^2 \alpha\pi} \right\} = -Z_1 U_1$$

$$E = 0, \qquad F = 0,$$

where $$Z_1 = \frac{PR^2(n+1)}{6\sqrt{3EI'Q}}.$$

In the wheel with non-radial spokes loaded in the plane of the rim the equation for the radial displacement u is *

$$\frac{\partial^6 u}{\partial \psi^6} + 2\frac{\partial^4 u}{\partial \psi^4} + (1 + K_1 \cos \epsilon)\frac{\partial^2 u}{\partial \psi^2} - K_1 u \sin \epsilon \tan \epsilon = 0$$

which gives

$u_r = D \cosh \gamma\psi + E \cos \alpha\psi \cosh \beta\psi + F \sin \alpha\psi \sinh \beta\psi,$

and

$u_t = A \sinh \gamma\psi + B \cos \alpha\psi \sinh \beta\psi + C \sin \alpha\psi \cosh \beta\psi,$

where u_r and u_t are the displacements when the load is radial and tangential respectively.

* Phil. Mag. xiv. p. 209 (1932).

Also * $u_r = \dfrac{\partial u_t}{\partial \psi}$.

Equation (7) for χ is the same form as that for u and K_2, and n in the former can be replaced by fictitious values of K_1 and ϵ which make the two equations identical so that α, β and γ have the same values. These fictitious values found by equating the coefficients of $\partial^2 u/\partial\psi^2$ and u to those of $\partial^2\chi/\partial\psi^2$ and χ respectively are given by

$$\left.\begin{aligned} \tan^2\epsilon &= n \\ K_1 &= K_2 \sec\epsilon \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Comparison of the constants D, E and F with G, H and U shows that

$$\frac{D}{G} = \frac{E}{H} = \frac{F}{U} = \frac{Z}{Z_1} = \frac{R}{n+1} . \quad . \quad . \quad . \quad (12)$$

Hence,

$$\chi = \frac{n+1}{R} u_r .$$

If the values of χ and $\partial^2 w/\partial\psi^2$ from equations (9) and (10) are substituted in equation (4) we obtain

$$N = \frac{EI'}{R^2}\left(u_r + \frac{\partial^2 u_r}{\partial\psi^2}\right),$$

which is the equation for the bending moment in the rim when a radial load acts in the plane of the wheel *.

Substitution for χ and w in equation (5) gives

$$\frac{\partial T}{\partial\psi} = \frac{EI'}{R^2}\left(u_r + \frac{\partial^2 u_r}{\partial\psi^2}\right).$$

Putting $u_r = \partial u_t/\partial\psi$ and integrating we have

$$T = \frac{EI'}{R^2}\left(u_t + \frac{\partial\, u_t}{\partial\psi^2}\right),$$

which is the equation for the bending moment in the rim when a tangential load acts in the plane of the wheel *.

Thus the distribution of N/PR round the rim and the maximum values can be found from the curves relating to M/PR for a radial load given in figs. 8 and 9 of the paper previously cited * using the fictitious values of K_1 and ϵ given by equation (11). Similarly the curves relating to M/PR for a tangential load in figs. 14, 17, and 16, of the same paper can be used for finding the variation round the rim

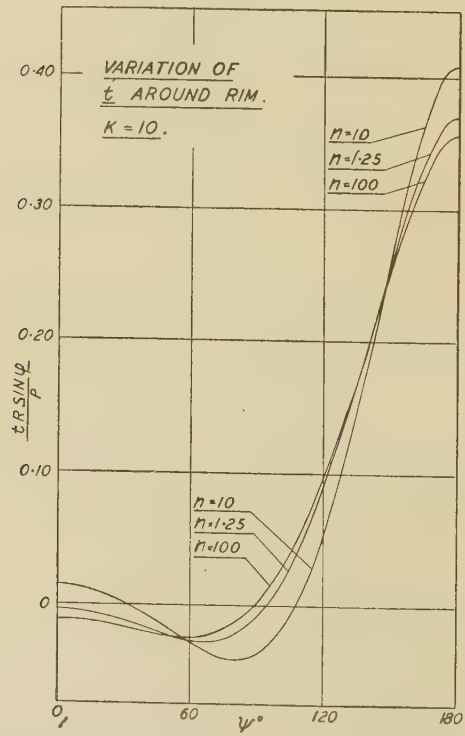
* Phil. Mag. xiv. p. 209 (1932).

of T/PR, the maximum values of this quantity, and the position of these maxima.

Variation of t around rim.

Since
$$\chi = \frac{n+1}{R} u_r,$$

Fig. 3.



we obtain from equation (8)
$$\frac{\partial^2 w}{\partial \psi^2} = n u_r - \frac{\partial^2 u_r}{\partial \psi^2}.$$

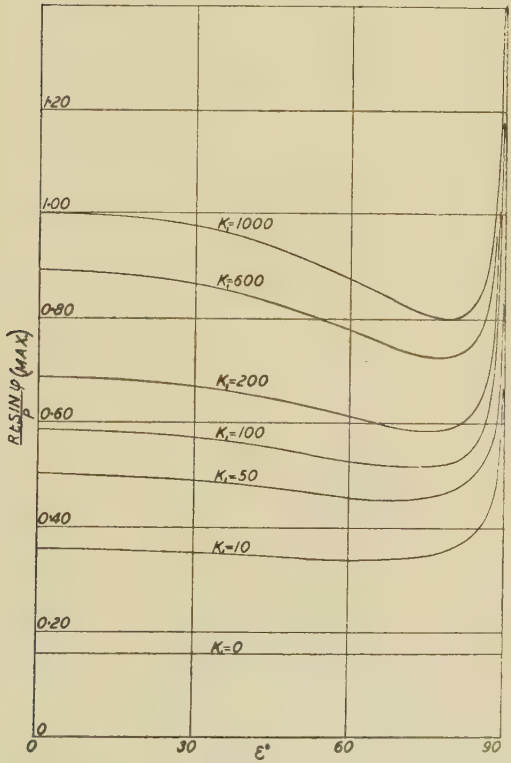
Integrating and substituting the relations between the values of u and v for tangential and radial loads * we obtain
$$w = -(u_r + n v_t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

* Phil. Mag. xiv. p. 209 (1932).

and either this or direct use of equation (10) gives the value of t at any point on the rim as

$$\frac{Rt \sin \phi}{P} = \frac{K_2}{12\sqrt{3Q}}\left[G_1 \frac{n-\gamma^2}{\gamma^2}\cosh \gamma\psi - \frac{H_1}{(\alpha^2+\beta^2)^2}\right\} 2\alpha\beta n \sin \alpha\psi \sinh \beta\psi$$

Fig. 4.



$$- \left(\overline{\alpha^2+\beta^2+n.\alpha^2-\beta^2}^2\right)\cos \alpha\psi \cosh \beta\psi\Big\}$$

$$- \frac{U_1}{(\alpha^2+\beta^2)^2}\Big\{ 2\alpha\beta n \cos \alpha\psi \cosh \beta\psi$$

$$+ \left(\alpha^2+\beta^2+n.\overline{\alpha^2-\beta^2}^2\right)\sin \alpha\psi \sinh \beta\psi\Big\}\Big] \quad . \quad (14)$$

Curves plotted in fig. 3 show the variation of t round the rim in certain selected cases.

TABLE I.

Values of $\frac{Rt \sin \phi}{P}$ (maximum).

K_1.	$\epsilon°$.										
	0.	10.	20.	30.	45.	60.	75.	85.	89° 59′.	90°.
0	0·1591	0·1591	0·1591	0·1591	0·1591	0·1591	0·1591	0·1591	0·1591	–
10	0·3567	0·3574	0·3540	0·3509	0·3445	0·3395	0·3486	0·3959	0·9504	∞
50	0·5010	0·5001	0·4958	0·4892	0·4751	0·4597	0·4568	0·5067	1·2428	∞
100	0·5858	0·5833	0·5784	0·5702	0·5515	0·5284	0·5163	0·5683	1·3950	∞
200	0·6874	0·6853	0·6789	0·6700	0·6435	0·6114	0·5873	0·6350	1·5658	∞
600	0·8924	0·8894	0·8801	0·8645	0·8291	0·7791	0·7282	0·7645	1·8805	∞
1000	1·0095	0·9983	0·9900	0·9777	0·9357	0·8754	0·8083	0·8356	2·0475	∞

Differentiation of equation (13) shows that the maximum value of w occurs at $\psi=\pi$ and this maximum is given by

$$\frac{\mathrm{R}t\sin\phi}{\mathrm{P}}=\frac{\mathrm{K}_2}{12\sqrt{3\mathrm{Q}}}\left[\frac{2(n-\gamma^2)\alpha\beta\cosh\gamma\pi}{\gamma}\right.$$
$$\left.+\frac{\alpha\{(\alpha^2+\beta^2)(3\alpha^2-\beta^2-2)+n(3\alpha^2-5\beta^2-2)\}\sinh 2\beta\pi}{+\beta\{(\alpha^2+\beta^2)(3\beta^2-\alpha^2+2)-n(3\beta^2-5\alpha^2+2)\}\sin 2\alpha\pi}\right]$$
$$(\alpha^2+\beta^2)\ (\cosh 2\beta\pi-\cos 2\alpha\pi)$$
$$\ldots\ldots\quad(15)$$

The curves of maximum values of $\frac{\mathrm{R}t}{\mathrm{P}}\sin\phi$ given in fig. 4 are plotted for convenience with the fictitious values of K_1 and ϵ as parameters instead of the real values K_2 and n. The results of this analysis can be applied to the case of a wheel with a finite but large number of spokes in the same manner as described in earlier papers.

The authors desire to thank the Aeronautical Research Committee for permission to publish this paper.

May 1931.

L. *Note on the most Probable Value of the Stefan-Boltzmann Radiation Constant.* *By* F. E. Hoare, *Ph.D., A.R.C.S.**

ALTHOUGH there have been a large number of determinations by widely differing methods of the Stefan-Boltzmann radiation constant, the available data are extremely discordant. This discordancy is shown by the listed results in Table I.

Attempts have been made by various workers in this field to improve the consistency of the data by the application of corrections for the lack of blackness of receivers, atmospheric absorption, etc. It appears to the present writer that there is great danger in such procedure, for, apart from the fact that the published experimental data are seldom sufficiently detailed for such corrections to be applied with confidence, there will, quite unconsciously, be a bias in favour of a correction which will bring the results of other investigators into substantial agreement with one's own.

* Communicated by Prof. F. H. Newman, D.Sc.

What is probably a more logical procedure, and one which is not so likely to be unconsciously influenced by a particular prejudice, is to assign weights to the individual determinations according to some fixed scale and to take the weighted mean.

In arriving at the weights to be assigned to the various determinations the following points have, as far as possible, been kept in mind :—

(1) Design of radiator.

(2) Design of receiver.

(3) Efficiency of receiving apparatus, *i. e.*, whether a good or poor absorber and method of determining correction.

(4) Method of operation of water-cooled shutters.

(5) Method of operation of receiver, *i. e.*, whether symmetrical or unsymmetrical.

(6) Accuracy possible in subsidiary measurements such as distance between source and receiver, area of apertures, electrical measurements, etc.

(7) Range of conditions over which observations were taken.

(8) Number of results obtained and their individual variation.

The above list is not exhaustive, but it is suggested that the major points influencing a determination are included, and that it makes a satisfactorily definite basis for the assessment of weights.

Naturally two investigators using the same table for their assessment would be unlikely to achieve complete agreement, as personal opinion regarding the relative merits of a particular investigation under each of the headings of a list, such as the above, will vary. Purely personal predilections will, however, be partially eliminated by assigning equal weight to each of the various headings, if necessary grouping two or more headings together so that this is justifiable, and then fixing the maximum weight to be given to any single determination. Also the divisibility of the weights must be fixed and the final weight brought to the nearest integral multiple of the unit. Thus, in assigning the weights given in Table I. each of the headings in the foregoing list has been given equal weight, with a maximum possible weight of 4 for any investigation.

Table I.

Note.—The numerical value of the Stefan-Boltzmann constant is of the order $5{\cdot}7 \times 10^{-5}$ erg/cm.^{2}/sec./degree^4. In what follows, only the significant figures will be mentioned, *e. g.*,

$$\sigma = 5{\cdot}735.$$

References *.	Observer.	Method.	Result.	Weight.
(*a*)	Kurlbaum.	Bolometer.	5·45	½
(*b*)	Fery.	Thermoelectric receiver.	6·30	½
(*c*)	Bauer and Moulin.	Fery pyrometer.	5·3	½
(*d*)	Bauer and Moulin.	Solar radiation.	5·7	1
(*e*)	Todd.	Thermal conductivity.	5·43	½
(*f*)	Valentiner.	Bolometer.	5·58	1
(*g*)	Fery and Drecq.	Thermoscope.	6·51	0
(*h*)	Fery and Drecq.	Fery pyrometer.	6·20	½
(*i*)	Shakespear.	Ratio of emissivities.	5·67	2
(*j*)	Puccianti.	Bolometer.	5·96	1
(*k*)	,,	Thermoscope.	6·15	1
(*l*)	Gerlach.	Modified Ångstrom pyrheliometer.	5·80	2
(*m*)	Westphal.	Ratio of emissivities.	5·54	2
	,,	,, ,,	5·57	
(*n*)	Keene.	Thermoscope.	5·89	1
(*o*)	Coblentz.	Modified Angstrom pyrheliometer.	5·722	3
(*p*)	Kahnowicz.	Modified Ångstrom pyrheliometer.	5·61	2
	,,	Modified Ångstrom pyrheliometer.	5·61	
(*q*)	Hoffmann.	Ratio of emissivities.	5·764	2
(*r*)	Kussmann.	Modified Ångstrom pyrheliometer.	5·795	2
(*s*)	Hoare.	Radio balance.	5·735	3½
	,,	,,	5·737	
(*t*)	Mendenhall.	Emissivity.	5·79	1

* See end of paper.

Table I. gives a summary of the results obtained up to the present time by different investigators. No investigation before that of Kurlbaum in 1898 is considered, as this is probably the earliest which can be regarded as at all comparable in accuracy with more modern work. If in a subsequent communication a correction has been applied by an investigator to his own previously published value, the latest revised value is entered in the table. In the case of the same investigator making observations by different methods the results from each are entered. The weights assigned by the method outlined above are given in the last column.

Taking the weighted mean of the results given in the foregoing table we obtain the value of the Stefan-Boltzmann constant as

$$5{\cdot}742 \pm 0{\cdot}027,$$

the probable error having been obtained by the usual formula for that of a weighted mean.

Birge (Phys. Rev. Suppl. i. p. 1 (1929)) discusses the data for the value of this constant then available, which includes all the determinations here listed except that made by Mendenhall and that of Hoare at the lower temperatures, and places the most probable value at

$$5{\cdot}735 \pm 0{\cdot}011.$$

The probable error of this value appears to be too low, but on the other hand, in view of the close agreement of the results obtained by the author over a wide range of conditions with the radio balance, the P.E. of the weighted mean given above is probably in excess of that required to represent fairly the most recent work on this constant. In the author's opinion the value

$$(5{\cdot}742 \pm 0{\cdot}019) \times 10^{-5}\ \text{erg/cm.}^2\text{/sec./degree}^4 \quad . \quad (1)$$

best represents the present state of our knowledge of the value of this constant.

An idea of the reliability of the value of the Stefan-Boltzmann constant thus found as a weighted mean may be obtained by using it in conjunction with Planck's radiation formula to obtain a value of h, Planck's constant.

Thus, taking the values of the velocity of light, c, and Boltzmann's constant, k, given by Birge (*loc. cit.*) as the most probable in conjunction with the value of the Stefan-Boltzmann constant, σ, given above (1), the value of h obtained by substitution in the equation

$$h = (2\pi^5 k^4 / 15 c^2 \sigma)^{1/3}$$

is found to be

$$(6{\cdot}536 \pm 0{\cdot}011) \times 10^{-27} \text{ erg.sec.}$$

This value of h is in good agreement with the values determined by other methods, as is shown by the following Table due to Birge :—

TABLE II.

Method.	h.
Rydberg constant	$6{\cdot}547 \pm 0{\cdot}011$
Ionization potentials	$6{\cdot}560 \pm 0{\cdot}015$
X-rays	$6{\cdot}550 \pm 0{\cdot}009$
Photoelectric	$6{\cdot}543 \pm 0{\cdot}010$
Constant of spectral radiation	$6{\cdot}548 \pm 0{\cdot}015$

References.

(*a*) *Ann. der Phys.* lxv. p. 746 (1898); *Verh. Phys. Gesell.* xiv. p. 576 (1912).
(*b*) *Comptes Rendus*, cxlviii. p. 915 (1909).
(*c*) *Comptes Rendus*, cxlix. p. 988 (1909); cl. p. 167 (1910).
(*d*) *Comptes Rendus*, cl. p. 1658 (1910).
(*e*) Proc. Roy. Soc. A, lxxxiii. p. 19 (1910).
(*f*) *Ann. der Phys.* xxxi. p. 275 (1910); xxxix. p. 489 (1912).
(*g*) *Comptes Rendus*, clii. p. 590 (1911).
(*h*) *Comptes Rendus*, clv. p. 1239 (1912).
(*i*) Proc. Roy. Soc. A, lxxxvi. p. 180 (1912).
(*j*) *Nuovo Cimento*, iv. p. 31 (1912).
(*k*) *Nuovo Cimento*, iv. p. 322 (1912).
(*l*) *Ann. der Phys.* xxxviii. p. 1 (1912); l. p. 259 (1916); *Zeit. für Phys.* ii. p. 76 (1920).
(*m*) *Verh. der Deutsch. Phys. Gesell.* xiv. p. 987 (1912); xv. p. 897 (1913).
(*n*) Proc. Roy. Soc. A, lxxxviii. p. 49 (1913).
(*o*) Bull. Bur. Stds. xii. p. 504 (1916); Proc. Nat. Acad. Sci. iii. p. 504 (1917).
(*p*) *Nuovo Cimento*, xiii. p. 142 (1917); *Atti d. Accad. Lin.* xxviii p. 73 (1919).
(*q*) *Zeits. für Phys.* xiv. p. 301 (1923).
(*r*) *Zeits. für Phys.* xxv. p. 58 (1924).
(*s*) Phil. Mag. vi. p. 828 (1928); xiii. p. 380 (1932).
(*t*) Phys. Rev. xxxiv. p. 502 (1929).

LI. *Some Experiments on Potassium Ions in Potassium Vapour.* *By* H. J. J. Braddick, *Ph.D.* (*Cantab.*), *Department of Physics, Trinity College, Dublin* *.

COLLISIONS between positive ions and atoms have been studied less exhaustively than collisions of electrons. In a paper by Ditchburn and Arnot [1] evidence was produced for the attachment of K^+ ions to potassium atoms and for the dissociation of the molecular ions thus produced when they pass through a magnetic field.

The present series of experiments was begun in order to examine the dissociation process, but it was not found possible to produce molecular ions by the process described by Ditchburn and Arnot. Some other ionization processes in potassium vapour were, however, examined.

In Ditchburn and Arnot's work it was found that the passage of K^+ ions through potassium vapour led to the formation of K_2^+ ions by attachment. It was assumed that momentum was conserved in this process, and a simple calculation shows that the kinetic energy of the K_2^+ molecule would be half that of the original positive ion, the balance of energy being converted into radiation. The probability was found to be quite high, the "effective atomic radius" for the process being about $4 \cdot 10^{-8}$ cm. This probability was nearly independent of the velocity of the ions between 20 and 200 volts.

In order to examine the process by an entirely different method it was proposed to accelerate potassium positive ions into a jet of potassium vapour and to examine the energy distribution of the emergent ions, so as to detect groups of particles which had suffered definite energy losses.

The Retarding Potential Apparatus.

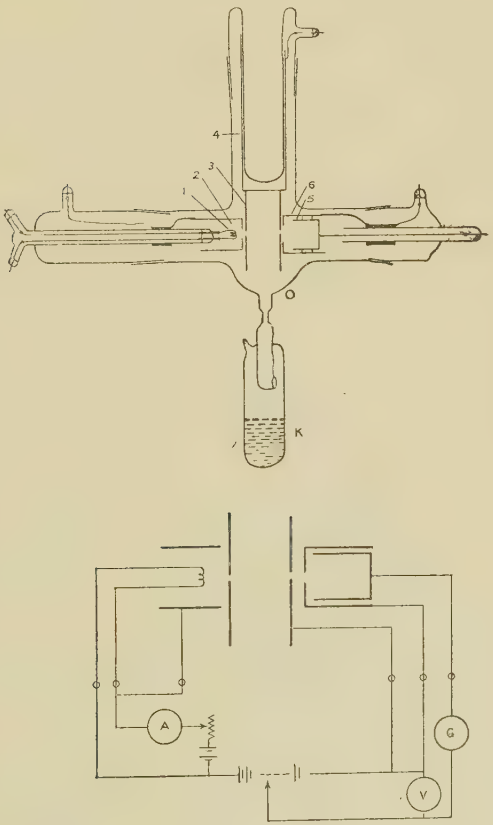
The first apparatus was constructed to carry out a retarding potential analysis of the emergent ions. It is represented in fig. 1. The potassium positive ions were obtained by heating the filament 1. This was a small coil of 250 μ tungsten wire covered with a Kunsman mixture prepared by mixing finely ground black iron oxide with 1–2 per cent. of potassium hydroxide, sintering the mixture in a blowpipe flame, and grinding it in an agate mortar. It was applied to the filament in melted paraffin wax, and the paraffin was burnt off by heating the filament in air. It was

* Communicated by the Author.

subsequently found (see p. 458) that these filaments gave K^+ ions almost free from any others.

The filament was surrounded by a metal shield 2, connected to the filament circuit in order to overcome fluctuations which had been traced to the effect of charges on the glass walls of the tube.

Fig. 1.



The accelerating potential was applied between the filament and the box (3), and the retarding (analyzing) potential between the box (3) and the Faraday cylinder (5).

The central electrode was welded to a thin copper tube which was sprung over a re-entrant glass tube 4. The tube 4 was kept filled with liquid air during a run, so that

the central electrode was kept cool and served to condense any potassium vapour that fell on it. The potassium vapour was supplied from a boiler K which contained the metal. The tube leading from K was constricted to form a rough slit, which was kept free from condensation and superheated by a little furnace-coil wound over it. K itself was heated by a furnace made from a piece of copper tube non-inductively wound and heated from A.C. mains. The whole apparatus was constructed of Pyrex glass and connected to a pumping system.

A number of small adjustments and alterations were made in the apparatus before it worked steadily and gave a satisfactory retarding potential curve. In the retarding potential curve finally obtained less than 1 per cent. of the particles were stopped by a retarding field 6 volts less than the accelerating potential.

A set of glass retorts was now fused on to boiler K and a charge of potassium introduced. The method of dealing with the metal was that described by Clark [4]. The accelerating potential was set at 180 volts and a number of retarding potential curves were taken with the potassium distilling into the apparatus. Particular attention was paid to looking for a break at about 90 volts corresponding with the attachment process described by Ditchburn and Arnot, and for a break at 45 volts corresponding to attachment followed by dissociation. No such breaks were found in a most careful examination with the potassium boiler at a temperature of 350°, though the probability of a collision between a potassium ion and a vapour atom was almost certainly at least as high as in Ditchburn and Arnot's work.

The difficuties in the use of the retarding potential method for velocity analysis has been pointed out by a number of workers*. In the present case the detection of a small group of particles with a particular energy requires a number of observations sufficient to determine the corresponding part of the retarding potential curve; and since it was found difficult to avoid small random fluctuations in the current the accuracy is not very high. A number of particles corresponding with those found by Ditchburn and Arnot could, however, have been detected with ease.

The Electrostatic Focussing Apparatus.

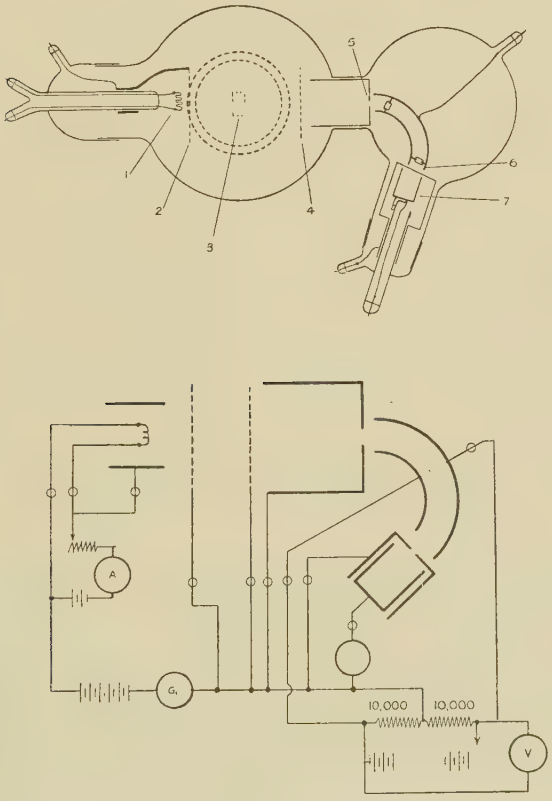
This apparatus gave directly an "energy spectrum" of the particle which had passed through the vapour. It has

* *E. g.*, Sharman, Camb. Phil. Soc. Proc. xxiii. p. 523.

been shown by Hughes and Rojansky [5] that a radial electrostatic field may be used in the velocity analysis of an electron beam, and that it has a refocussing effect if the deflexion of the beam is about 127°.

The apparatus constructed to use this principle in the analysis of the potassium ions is shown in fig. 2. The

Fig. 2.



positive ions were produced from a Kunsman filament (1), and accelerated through a gauze (2) into the stream of potassium vapour. The potassium stream was obtained from a boiler underneath the main bulb, passed upwards through a slit (3) in a copper plate, and condensed mainly on a re-entrant tube cooled by liquid air. The ions passed through another gauze (4) to the first slit (5) of the analyzing chamber, and were then deflected by the electrostatic field. The

second slit (6) of the analyzer was cut in the plane front of a large copper shielding cylinder which contained the Faraday cylinder. The deflecting plates were bent from stout sheet copper, and copper lugs were attached by silver solder. These lugs were fused into little lumps of Pyrex glass, so that the two plates were held together rigidly.

A first attempt was made to use a galvanometer (Leeds-Northrup; sensitivity 10^{-10} amps. per mm.) to detect the collected current, but the sensitivity was not sufficient, since the main beam of K^+ ions only gave a deflexion of about 10 mm.

The cylinder was then connected to a Compton electrometer used at low sensitivity, and the current was read by observing the rate of deflexion. Since only comparative measures of current were required the current sensitivity of this arrangement was not measured, but it was of the order 10^{-13} amp. for 1 mm./sec.

The apparatus worked satisfactorily. When the accelerating potential was held at 160 v. and the potassium was left cold the current was much less than 1/1000 of the main peak everywhere except in the region $V_a = 100$ v., when there was a tiny hump of the order 1/300—this seemed to be a peculiarity of the apparatus, and was not further investigated.

The accelerating potential V_a and the deflecting potential V_d were accurately proportional over the range 10–300 volts.

The accelerating potential was set at 160 volts (the deflecting potential corresponding was 120 volts) and a careful search was made for particles coming in at deflecting potential about 60 volts. The electrometer was watched carefully, while the potential on the deflecting plates was varied continuously by a potentiometer connected across part of the battery. No peak of current was detected in this region, though the conditions were such that a current corresponding to 1/2000 of the main beam of particles could probably have been detected, and a current of 1/1000 of the main beam could not have escaped notice.

During the experiment the temperature of the boiler was 290° and the potassium was distilling freely into the bulb.

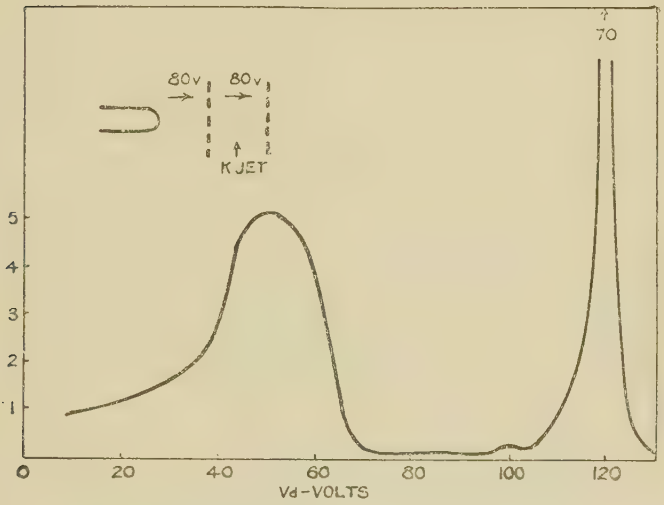
No "half energy" ions were found in experiments in which the accelerating potential was 80 volts, 190 volts, or 270 volts.

In these experiments a small diffuse peak was obtained at low deflecting voltages when the potassium vapour was present, and disappeared when the potassium bulb was allowed to cool. This peak was not due to positive ions which had

collided with gas molecules accidentally present, for the distribution of velocities was entirely different from that observed when the vacuum was made bad by shutting off the diffusion pump.

In this apparatus no precaution was taken to prevent the potassium vapour from getting into the space between the Kunsman filament and the first grid, the region where the ions were accelerated. The small humps are probably due to the ionization of the potassium in this region by processes which will be discussed below.

Fig. 3.



The arrangement of the accelerating potentials was now altered so that the ions were accelerated by a potential of 80 volts to the first gauze, 2 (fig. 3). They then passed through the space where the greatest concentration of potassium vapour was maintained, and ions were drawn away from this region by a potential of 80 volts applied between gauzes 2 and 4. With the potassium cold there was only one peak corresponding to the ions which had fallen through the total potential of 160 volts, but when the potassium vapour was distilling into the apparatus there appeared, in addition, a diffuse peak as indicated in the diagram (fig. 3). The magnitude of this peak rose rapidly as the furnace temperature was raised, and though the apparatus was entirely

unsuited to quantitative measurements of the number of particles forming the peak, the number appeared to be comparable with that in the main peak when the potassium boiler was maintained at 250°.

An examination of fig. 3 indicates that the peak corresponds with ions which are produced in the jet of vapour with small or zero velocities. In another set of experiments the potentials applied to the apparatus were altered, and in every case the position of the "low velocity" peak was in accordance with this assumption.

Such ions might be formed by secondary processes, in particular by the ionization of the vapour by electrons liberated from the gauzes by the impact of positive ions. It is known that a contaminated metal surface gives off secondary electrons when bombarded with K ions[8], and these would be accelerated in the electric field in the reverse direction to the potassium ions and might produce ionization in the potassium jet. In order to see if the observed ionization could be explained in this way a large coil was set up at the side of the bulb, so that a magnetic field of about 75 gauss could be established over the central part of the apparatus. This field was sufficient to confine the secondary electrons to the immediate neighbourhood of the gauzes. The number and distribution of the low-velocity particles were not appreciably changed by switching the current on in the coil.

This result is not quite conclusive, but it suggests very strongly that the slow-moving particles are produced by direct collision of the positive ions and not by secondary electron emission.

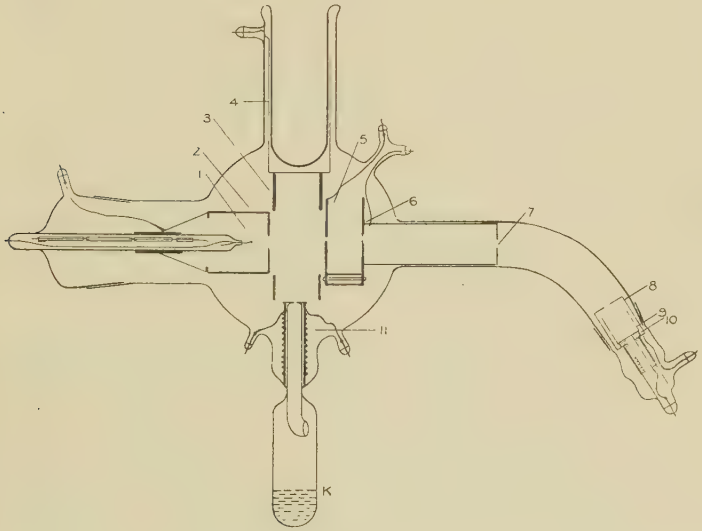
The ions may be produced by the process of "Umladung" described by Kallmann and Rosen[7]. They found that ions (*e. g.*, of Ar and Ne) show a strong selective absorption in their own gas, and this they ascribe to a process in which a rapidly moving ion A^+ undergoes a collision with a neutral atom B, and receives an electron from it without much transfer of kinetic energy. The products of this collision are a slowly moving ion B^+ and a rapidly moving neutral particle A. According to Kallmann and Rosen the probability of the process is high when A and B are similar atoms.

The Magnetic Deflexion Apparatus.

An apparatus was constructed in which the ions were separated by magnetic deflexion as in Ditchburn and Arnot's work, and this is represented in fig. 4. The Kunsman filament is here shown enclosed by the copper cylinder (2),

which forms a kind of potassium ion gun. In some of the experiments, however, the filament was moved forward into the copper shield (3), and was therefore exposed to the potassium jet. The potassium vapour was obtained from a boiler K heated in an electric furnace, and it emerged into the apparatus from a wide slit (4 mm. × 9 mm.) cut in a molybdenum cup. The tube (11) was kept superheated by a small furnace. The potassium jet was fired into the box (3) which was supported from a re-entrant tube (4) cooled with liquid air. It was found, however, during some

Fig. 4.



of the experiments described below that this arrangement was not sufficient to confine all the potassium vapour to the central part of the apparatus.

A further accelerating field could be applied to the ions after they had passed through the box (3), but in order to avoid difficulties of aligning the electrode (3) accurately the main accelerating field was applied between the plates (5 and 6), which were made of stout sheet copper and held parallel by an arrangement of Pyrex spacing washers. The slits (6 and 7) defined a beam of particles which was deflected in the magnetic field and collected in the Faraday cylinder (9). The inside of the glass tube passing through the magnetic field was covered with two layers of chemically deposited

silver *, and the walls made good contact with the copper tube carrying the slits (6 and 7), which was sprung into place. The magnetic field was provided by a small Weiss electromagnet: the regulation and measurement of the magnetic field is described below. A magnetic field consisting of of a heavy iron pipe split longitudinally was fastened over the part of the tube between 6 and 7.

For most of the experiments with this apparatus the Compton electrometer was used as a constant deflexion instrument. The sensitivity was set at about 2000 divs. per volt., and the electrometer was shunted with a resistance made by drawing a pencil line on an amber rod, clamping copper wire tightly over the ends of the line, and sealing up in a glass tube with a lump of calcium chloride. The resistance was checked from time to time with a high-tension battery and a sensitive galvanometer. It was found to vary slowly with time but to keep very constant over intervals of a day or so. The least current readily measurable was 10^{-13} amp., giving a deflexion of 1 mm. In some cases, when currents of very different strengths were to be compared, the Leeds Northrup galvanometer of sensitivity of about 10^{-10} amp. per division was connected so that it was brought into action by lowering the electrometer earthing key.

In the use of this apparatus the magnet current was first increased to a maximum value and then gradually reduced, a given value being always approached from above. The field obtained in this way was fairly reproducible, and at the lower fields was proportional to the magnet current. When it was necessary to establish the position of a particular peak in the magnetic spectrum the mean magnetic field over the central portion of the pole pieces was measured by a flat circular search-coil connected to a table galvanometer.

In the first experiments with this apparatus the filament, coated with Kunsman mixture from the same batch as before, was mounted immediately in the vapour jet and within the electrode. The curve obtained in the absence of the potassium vapour showed only K^+ ions. When the potassium was heated a small peak was observed in the "high mass" region, in a position corresponding quite accurately with a singly charged particle of twice the mass of the K atom which had fallen through the full voltage applied to

* This was more satisfactory than a volatilized magnesium layer or a layer of sputtered platinum.

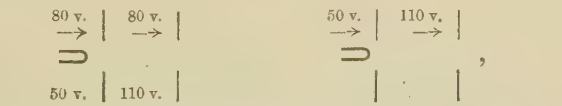
the tube. This small peak persisted after the furnace had been cooled with its intensity almost unchanged. Its height was about 1/200 of that of the main maximum corresponding to K^+.

Since this effect might be due to the retention of metallic potassium on the filament, a tungsten filament was coated with iron oxide without any addition of potassium salt. This filament gave an emission consisting largely of potassium ions which gradually ceased when the filament was heated for some hours in vacuum. When the potassium vapour jet was put on there was a large emission of K^+ ions (order 10^{-7} amp.) and a small peak corresponding to K_2^+ was also observed. When the potassium jet was stopped by cooling the boiler the emission persisted without much decrease for many hours with the filament at a temperature in the neighbourhood of 900°. When the filament was glowed more brightly the K^+ emission fell off with time, but the K_2^+ emission peak remained always at about 1/3000 of the K peak.

In order to verify that the peak was due to ions from the filament the potentials applied to the electrodes were varied, and the following results were obtained :—

(1) The total accelerating potential was kept constant, but it was differently divided between the filament F and the plates of the accelerating condenser.

The arrangements used were



and with all of these the peak remained unchanged in position.

(2) The total accelerating potential was altered, the peaks were again observed at the appropriate magnet currents.

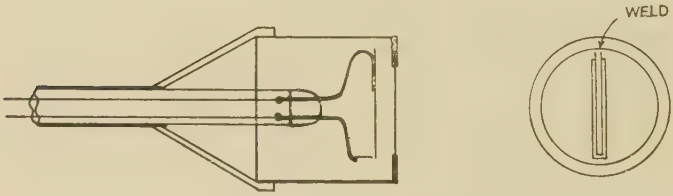
In all these experiments the peak at mass no. 19·5 was looked for. This peak was reported by Ditchburn and Arnot, and ascribed by them to the dissociation of the K_2^+ ions. It was not observed in any of the present experiments.

The apparatus was now modified by withdrawing the filament into the position shown in fig. 4, where it was shielded from the potassium vapour. The conditions were now similar to those contemplated by the explanation of Ditchburn and Arnot (the potassium ions were accelerated

into a jet of potassium vapour and analyzed after passage through the vapour). In these experiments no K_2^+ ions were detected, though it is certain that K_2^+ ions to the extent of 1/1000 of the K^+ emission could have been found. In these experiments a powerful subsidiary peak was found on the low velocity side of the main peak due to K^+ ions, and when the potentials applied to the different electrodes were varied this peak moved in a way which showed that it was due to K^+ ions produced in the jet of vapour with small or no velocity and then accelerated in the electric field. The explanation of this peak is clearly the same as for that encountered in the electrostatic experiments and given on p. 456.

During all the work which has been described there had been considerable trouble with capricious variations in the current collected by the electrometer circuit, even when the

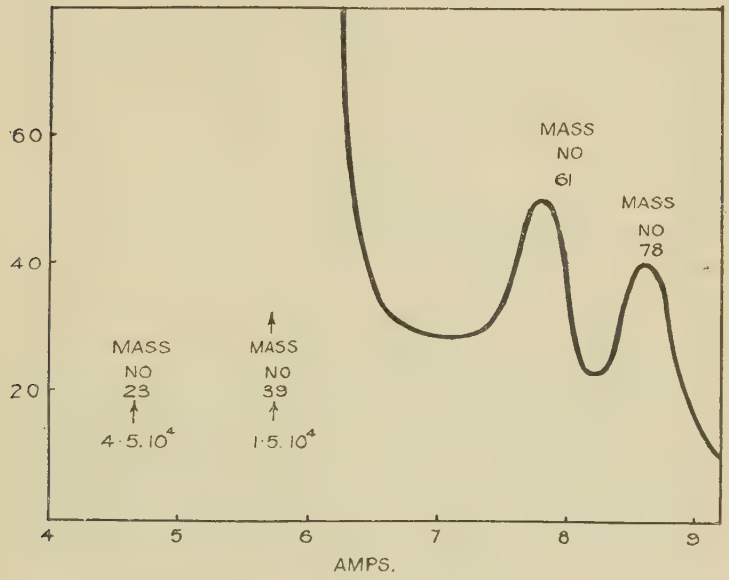
Fig. 5.



ion current to the gun casing (taken as a measure of the emission) and all the other conditions were kept steady. This was attributed to the emission of directed jets of ions from the heated Kunsman mixture (Dr. E. T. S. Walton told me that this effect was also observed in his experiments with Kunsman sources), and at the end of this series of experiments a form of Kunsman source was developed which seemed much better in this respect: this is shown in fig. 5. In order to liberate the ions into a uniform electric field, the emitting mixture was spread over a tungsten strip which was let into a slit in a molybdenum plate. This plate was fixed inside a cylindrical "gun," with its plane parallel to the front of the latter. The ions emerged from the gun through a wide slit covered by a grating of fine wires. In the final form of this arrangement a tungsten strip 1 mm. × 0·2 mm. was overwound with a spiral of 0·1 molybdenum, which retained the Kunsman mixture in place and was mounted as described. A very steady positive ion current was obtained from it. A filament coated with

sodium Kunsman mixture was arranged so that Na^+ ions could be accelerated into a potassium beam, as it was hoped to investigate the exchange of charge between Na^+ ions and K atoms. It was, however, found impossible to shield the emitter in this arrangement from the potassium vapour. After a few minutes exposure to the potassium vapour the filament emitted K^+ ions very much more strongly than Na^+ ions. At the same time K_2^+ ions appeared to the extent of about 1/2000 of the K^+ emission. In some of the experiments a small peak was observed at a magnetic field

Fig. 6.



(as measured by a fluxmeter) corresponding to the molecular ion $K\,Na^+$, but the conditions in which this was formed could not be established with any certainty. A curve showing the peak ascribed to $K\,Na^+$ is shown in fig. 6. The magnetic field as measured by the fluxmeter corresponded fairly accurately with the mass no. 62.

An emitter coated with a potassium mixture was also used in this apparatus ; after exposure to the action of potassium vapour this gave an emission of K_2^+ ions about 1/300 of the emission of K^+ ions. Traces of Na^+ were detected, and in some of the experiments there was a marked peak corresponding to the mass no. 32. The origin of this is not

certain, but it may be due to O_2^+ as obtained by Murawkin [5] in the positive ion emission from certain glasses. The ion O^+ could not be found.

Finally, experiments were conducted with a tungsten strip emitter without a Kunsman mixture.

The tungsten strip was glowed at a high temperature until the initial evolution of ions had almost ceased. It was then exposed to the potassium vapour, and while the potassium jet was on there was a copious emission of K^+ ions and K_2^+ ions. After the potassium boiler had been cooled the ion current persisted for a long time, falling off over a period of some hours with the filament at about 900° C. This plain filament gave during this time the largest proportional emission of K_2^+ ions which was recorded. The emission of K_2^+ ions was about $1\frac{1}{2}$ per cent. of the emission of K^+ ions, and the ratio

$$\frac{K_2^+ \text{ emission current}}{K^+ \text{ emission current}}$$

did not change significantly when temperature of the filament was increased or decreased.

Discussion of Results.

It has been shown by the retarding potential and electrostatic deflexion experiments that the process described by Ditchburn and Arnot—*i.e.*, the attachment of a relatively fast moving potassium ion to a potassium atom—does not take place, or, at any rate, that it does not take place with sufficient frequency to explain their results, though these experiments do not exclude the possibility that potassium ions moving with energies of only a few volts may undergo attachments.

It is clear that the process

$$K^+ + K \rightarrow K_2^+$$

occurring to a fast moving ion would be accompanied by a large disappearance of kinetic energy. The liberation of so large a quantity of radiation (of the order of 100 electron volts) would be difficult to explain in the circumstances. Much larger quanta must be liberated in an elementary act in the production of "white" X-rays by electron impact, but experiments on the production of X-rays by positive ion impact have failed to give positive results.

In the present series of experiments the ion K_2^+ was detected in the experiments made with the magnetic deflexion apparatus when the Kunsman filament was exposed

to potassium vapour and also when a plain tungsten filament was exposed. The small peak on the "high mass" side of the principal maximum agrees quite accurately with the assumption of K_2^+ ions if these fall through the whole potential difference applied to the tube, *i.e.*, if they are formed on the surface of the filament or before the ions have been accelerated through more than a few volts. These ions were also found after the potassium jet had been cut off, and this suggests strongly that they are formed from a layer of potassium deposited on the filament.

It may be noted that in the emission from the Kunsman mixture which had not been exposed to the potassium vapour only *atomic* ions were detected, and this would be expected if the potassium ions from a Kunsman source are extracted electrolytically from an ionic lattice.

When emission takes place from an adsorbed film of metallic potassium some molecular ions are, however, formed. These might be produced

(1) by the direct evaporation of molecular ions from molecules forming part of the surface film;

(2) by an attachment process taking place between an atomic ion and an atom in the immediate neighbourhood of the surface.

Both these explanations offer difficulties in explaining the observations that the ratio K_2^+/K^+ does not vary much with the temperature of the filament or the strength of the potassium jet. Lewis [11] has shown that in potassium vapour escaping from a furnace in which it is in equilibrium the ratio of molecules to atoms is about $1\frac{1}{2}$ per cent. at 600° K., and rises with the temperature on account of the increase in vapour pressure displacing the equilibrium

$$\frac{[K]^2}{K_2} = P_K.$$

At constant concentration the proportion of molecules would decrease with increasing temperature, since the heat of dissociation of K_2 is positive. In the case of the adsorbed film we may suppose that there is a similar equilibrium between atoms and molecules. In order to explain the observation that the ratio K_2^+/K^+ does not change rapidly with temperature we should have to assume that the relative rate of evaporation of K_2^+ and K^+ as ions changes with temperature in such a way as to balance the displacement of the equilibrium between K_2^+ and K, and this is possible

since the ionization potential of K is higher than that of K^+ and the evaporation with ionization of K would therefore be favoured by a rise in temperature.

The second explanation requires that the chance of a K^+ ion forming an attachment as it leaves the filament shall be independent of temperature. We may suppose that this attachment takes place substantially as explained in Ditchburn and Arnot's paper, but the conditions seem to be complicated by the presence of the surface, and a calculation of the probability of attachment is not possible without some picture of conditions at the surface. This second explanation is supported by the experiments in which NaK^+ molecular ions were observed in the emission from an Na Kunsman filament which had been exposed to K vapour, since the Na in this case is presumably emitted originally as atomic ions which then undergo attachment.

It must be emphasized that the experiments described here are entirely qualitative. It is hoped to make a more careful quantitative study of the variation of the emission of K_2^+ with temperature and with the state of the filament surface—in particular the amount of potassium on it—which may clear up the mechanism of the emission. It is almost certain that the K_2^+ ions reported by Ditchburn and Arnot were obtained by the same process as the ions discussed here. They only observed K_2^+ ions, however, in the presence of an actual jet of potassium vapour, and this is in disagreement with the present experiments.

In some experiments by Bainbridge [10] a tungsten filament was heated in a mixture of alkali metal vapours, and the ions were analyzed by a refocussing magnetic deflexion method. The only molecular ion found was Na_2^+, which was nearly as frequent as Na^+. As Bainbridge points out this result is anomalous and is certainly difficult to reconcile with the present experiments.

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LII. *Time-lag in Photo-cells and the Townsend Discharge.* *By* N. R. Campbell *. (Communication from the Staff of the Research Laboratories of The General Electric Company, Limited, Wembley, England.)

Summary.

The paper is a continuation of the work of Campbell and Stoodley (3).

The *Introduction* sketches the facts already known and the theories propounded to explain them. The problem is to relate the facts concerning time-lag, more complicated than is generally realized, to the theory of the Townsend discharge.

The *apparatus* was designed, (1) for taking "frequency response" curves, as previous workers have done, for light varying with known frequency, and (2) for determining the time-lag directly by following the passage of electricity through the cell after the incidence of a single flash of light on it. A pendulum, transmitting the flash and then interrupting the connexion between cell and electrometer, was used for this purpose. The chief features of the apparatus are described.

The chief *results* established are :—(1) The formation of charges on the walls is not the cause of the ordinary time-lag ; (2) there is a form of time-lag (called here L. F.), associated especially with "Thin Film" cathodes of low work-function, and probably arising in the production of secondary emission at them ; (3) in this time-lag the usual fall in the frequency-response curve with increasing applied voltage and magnification is due to an increase in the period of lag rather than in the fraction of the current affected by the lag ; (4) this lag is increased by fall of temperature and therefore probably arises in some thermal motion.

A *theory* is proposed which attributes this time-lag to the diffusion of metastable atoms to the cathode ; but it proves to be untenable. The difficulties of finding an alternative theory are discussed ; it is suggested that further knowledge of the processes occurring at the cathode in a Townsend discharge is required before progress can be made. A start in this direction is made by examining how the time-lag at constant magnification varies with the combination of gas-pressure and voltage producing that magnification, and by trying to explain the difference between characteristics of

* Communicated by C. C. Paterson, M.I.E.E.

similar cells with different cathodes in terms of secondary emission at the cathode associated with the production of positive ions.

I. *Introduction.*

1. IF a photoelectric cell is illuminated by interrupted light (*e. g.*, that passing through holes in a rotating disk) the current through it is compounded of a mean current $\bar{\imath}$ and of an alternating component of amplitude i_a. If the cell is evacuated or, being gas-filled, is used at a potential below that of ionization, both $\bar{\imath}$ and i_a are independent of n, the frequency of interruption. If there is ionization of the gas-filling $\bar{\imath}$ is still independent of the frequency, but i_a decreases as n increases.

This decrease of i_a is attributed to a *time-lag* which prevents the current rising to its full value immediately the light is turned on and causes it to continue after the light is turned off. It will be convenient if we express this theory mathematically at the outset. The fundamental assumption is that if light capable of producing a steady current $\bar{\imath}$ falls on the cell only during the short time $\delta\tau$ at $\tau=0$, then q, the charge that will have passed through the cell at any subsequent time τ, is given by

$$q=\bar{\imath}\delta\tau f(\tau/\tau_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where f increases with τ from $0<f(0)<1$ to $f(\infty)=1$. Among cells for which f has the same form τ_0 is a measure of the "*time-lag*."

In taking a *frequency-response curve* ideally we vary the light harmonically about its mean, so that if $i(t)$ is the steady current corresponding to light incident at t (if it remained constant)

$$i(t)=\bar{\imath}(e^{iwt}+1), \; w=2\pi n. \quad . \quad . \quad . \quad . \quad (2)$$

$i(t,\tau)$, the current flowing at time t in virtue of the light that was incident during $\delta\tau$ at time $t-\tau$, is given by

$$i(t,\tau)=i(t-\tau)\delta\tau.f'(\tau/\tau_0); \quad . \quad . \quad . \quad . \quad (3)$$

the whole current i is given by

$$i=\int_0 i(t,\tau)d\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It follows that the measured ratio of amplitude to mean is given by

$$i_a/\bar{\imath}=\int_0^\infty e^{-iw\tau}f'(\tau/\tau_0)d\tau. \quad . \quad . \quad . \quad . \quad (5)$$

It is probable *a priori*, and there is some actual evidence[3], that f has approximately the form

$$f(\tau/\tau_0) = 1 - ae^{-\lambda\tau}, \text{ where } \lambda = 1/\tau_0, \quad . \quad . \quad . \quad (6)$$

so that q consists of a fraction $(1-a)$ which passes instantaneously and a fraction a subject to time-lag. (In order to preserve the continuity of f at $\tau = 0$ the first fraction must be regarded as similar to the second, but with a much greater λ and smaller τ_0.) If so (5) becomes

$$\left(\frac{i_a}{\bar{i}}\right)^2 = \frac{(1-a).\omega^2+\lambda^2)^2 + a\omega^2\lambda^2}{(\omega^2+\lambda^2)^2}. \quad . \quad . \quad . \quad (7)$$

It will be seen that when λ is of the same order as ω a decrease of $i_a/\bar{i}$ at constant ω may be due either to a decrease of λ or to an increase of a—that is to say, either to an increase of real time-lag or to an increase of the fraction of the current affected by time-lag. Frequency response curves are ill suited to distinguish between these alternatives; for even if the variation of $i_a/\bar{i}$ with ω over a considerable range is known, the presence of two adjustable constants in (7), closely intermingled, makes the estimation of either of them very precarious, especially when (7) does not fit the observations exactly with any values of a and λ, and when there is therefore no evidence that f is exactly of the form (6).

2. The customary description of a decrease of $i_a/\bar{i}$ at constant ω as an increase in time-lag may therefore prove misleading; but if we adopt it provisionally the facts already established may be stated thus:—

(1) The time-lag increases steadily with the *magnification* m, which is the ratio of the steady current in a gas-filled cell at any voltage to the electronic emission that would be produced by the same light in the evacuated cell[1, 2, 3].

(2) The time-lag is generally greater at the same m if the gas is helium or neon than if it is argon or hydrogen[2, 3].

(3) But the time-lag depends on the nature of the cathode and the geometrical form of the cell and electrodes as well as on the gas[3].

3. The time-lag, when first discovered, was attributed to the finite time required for the positive ions to reach the electrodes; but calculation shows that this time is too short to account for the facts[3], unless possibly it is greatly prolonged by the liberation of secondary electrons from the cathode by the impinging ions[12]. The most popular theory

at present seems to be that of Schröter and Lubszynski [2], who attribute it to the formation of metastable atoms. These atoms endure for an appreciable time after their formation, and disappear finally by ionizing impurities in the gas ; they thus give rise to current persisting after the light ceases to act. The further suggestion that excited or metastable atoms of some kind may give rise to appreciable time-lag by being easily ionized at a second electron impact must be rejected ; for this suggestion, and every other that involves appreciable interaction between the effects of primary electrons not emitted simultaneously, is inconsistent with the well-established fact that the mean current $\bar{\imath}$ is totally independent of the frequency of interruption. In the experiments described below we have been careful to investigate this matter, and have satisfied ourselves that over the range of frequency from 5 to 8000 interruptions per second $\bar{\imath}$ certainly does not vary by as much as 0·3 per cent. even when i_a decreases by 50 per cent. The significance of this fact is sometimes overlooked.

A less definite theory is that of Fourmarier [4], who suggests that time-lag arises in the process of ionization by positive ions ; but the evidence adduced is very indirect, and there appears to be no other reason for believing that ionization by positive ions is more likely to be affected by time-lag than ionization by electrons. Lastly, there is the suggestion of Wheatcroft [5] that time-lag in photoelectric cells may be due to the same cause as the irregularities in the sparking potential in the intermittent glow-discharge. Rothe [6] has attributed these irregularities to the formation of double-layers of ions and electrons on the cathode, which endure for a finite time and then break up and pass to the appropriate electrodes. If such double-layers are formed in the glow-discharge they may also be formed in the Townsend discharge ; they might account for the well-known variability of the sparking potential of a photoelectric cell as well as for its time-lag. Another theory attributing time-lag to a layer of ions on the cathode has been proposed by Kingdon and Thomson [13].

These theories are not mutually inconsistent; for time-lag may be, and probably is, due to several causes. Thus there must be some time-lag of the kind indicated by Schröter and Lubszynski ; and yet without elaboration and amplification their theory can hardly account for the influence of the cathode. Before a more complete theory can be framed many more facts must be available. Most workers have based their conclusions on the examination of a very narrow range of cells. Our object at the outset was to extend the

work of Campbell and Stoodley [3] by examining a large number of cells of different types * ; but we soon reached a stage at which accumulation of data could serve no useful purpose until further information which would probably have to be derived from totally different experiments, was available to coordinate them. It seems to us of little use to investigate time-lag until more is known of the Townsend discharge ; the presentation of our reasons for this conclusion is the main object of our paper.

II. *Apparatus.*

4. In one part of our experiments we determined frequency-response curves with apparatus of the usual kind. It consisted of a rotating disk pierced with holes through which light fell on the cell ; the current from the cell was fed to an amplifier, the output of which was measured. Full details need not be given ; notes on the more important features will suffice.

(*a*) If the response curves were to be interpreted quantitatively it would be very important that the variation of the light should be truly simple-harmonic. Milne and Fowler [7] have shown that this condition is not easily fulfilled with a rotating disk ; a good approximation is obtained by simple means when the holes are circular, spaced at distances equal to their diameter, and pass over a slit whose width is equal to their diameter. With this arrangement, which was adopted, the relative amplitudes of the successive harmonics should be

1	1·0000
3	0·0558
5	0·0145
7	0·0062, etc.

Rough measurements with filters in the output circuit showed that values not far from these were actually attained; but there was some evidence of even harmonics, which should have been entirely absent. However, the approximation was clearly good enough not to invalidate such conclusions as we propose to draw.

(*b*) In order that error might not be introduced by electrical disturbances arising from the motor driving the disk,

* The work has been persistently hampered by ill-health. For this reason Mr. Stoodley, who was responsible for planning the research and developing most of the experimental methods, had to abandon the work, which was thereby greatly delayed. And now the illness of the survivor compels the presentation of the results in an incomplete form.

the disk was run up to full speed, the motor switched off, and as the disk slowed down a tachometer indicating its speed and the output voltmeter were read simultaneously. This method was much more convenient than that of attempting to maintain the disk at a constant speed, and introduced no serious error, except occasionally at very slow speeds, when the speed of the disk and the reading on the voltmeter were both changing rapidly. The result of a mean of three runs could always be reproduced within 0·5 per cent.

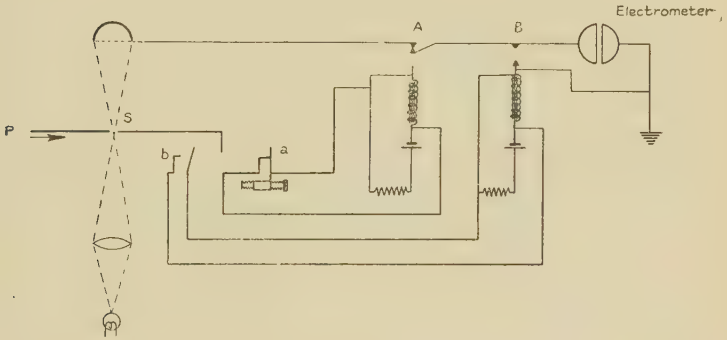
(*c*) The prime requisite of the amplifier is that the ratio of output to the amplitude of the photoelectric current should be determinable once and for all by calibration. A two-stage resistance-coupled amplifier was employed using Osram P 240 and P 2 valves. The input resistance, through which the photoelectric current passed, was 0·25 megohm ; since the current never exceeded 1μ A, the change in voltage across the cell during a cycle never exceeded 0·12 volt and never produced a change in the current of more than 1 per cent. The effective input capacity was about $30\,\mu\mu$ F, and did not vary by more than $3\,\mu\mu$ F with different cells ; the input impedance was therefore constant to 0·2 per cent. The output voltmeter was calibrated in the first instance by 50-cycle voltage applied to the input. It then appeared that with any cell and at any constant speed of the disk the output was accurately proportional to the mean current passing through the cell, measured on a galvanometer in the cell circuit. The amplification was therefore linear whatever the frequency or wave-form, so long as these were constant. The factor necessary to derive $i_a/\bar{\imath}$ from the measured ratio of the output volts to the galvanometer current was obtained by assuming that $i_a/\bar{\imath} = 1$ for a vacuum cell at all frequencies ; the factor varied by about 8 per cent. for frequencies between 0 and 8000 p.p.s.

5. For the reasons given in para. 1 we realized that frequency-response curves must be supplemented by some more direct method of determining $f(\tau/\tau_0)$. Since the ambiguity of these curves is partly due to the confusion of effects due to changes of amplitude with those due to changes of phase, Wheatcroft's method [5], which determines the phase, has some undoubted advantages. But no method could be as good as one that uses (1) directly and determines the events consequent on the incidence of a single flash of light on the cell. The natural instrument to use is the cathode-ray oscillograph, with which Campbell and Stoodley have already experimented [3] ; and there is no doubt that this is the most

powerful instrument for the purpose ; but it was found that somewhat elaborate arrangements would be required to attain the necessary sensitivity and accuracy, and it seemed desirable to make some experiments in which no ambiguity could be introduced by the use of amplifiers. Accordingly a method was adopted which, though less powerful, was free from all objections of principle ; it is shown diagrammatically in fig. 1.

The cell was connected to an electrometer through a switch A. A pendulum p, carrying a slit s, allowed a flash of light to fall on the cell and then broke a contact a, carried on a micrometer screw ; this contact operated a magnet which opened A. The time between the flash and the disconnection of the cell from the electrometer was thus

Fig. 1.



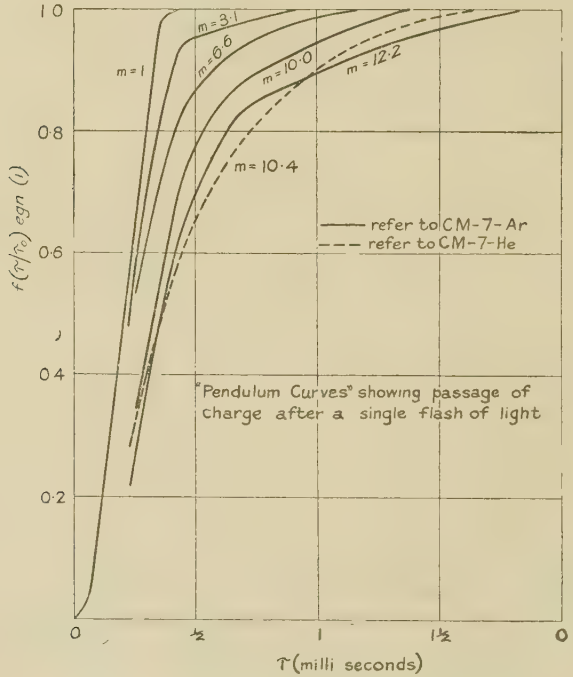
adjustable and measurable ; the charge received by the electrometer was that which had passed through the cell in this time. The switch B, insulating the electrometer, was opened by another contact b, worked by the pendulum just before the light passed through the slit. Typical results are shown by the curves of fig. 2, in which the charge is plotted against the time ; if the flash had been truly instantaneous such curves (called here "pendulum curves") would give $f(\tau/\tau_0)$ directly.

The apparatus needed careful design, but only a few features require specific mention. In order that the change of voltage across the cell should not affect its characteristic materially during the measurement, the rise of the electrometer voltage had to be limited to 0·25 volt ; an instrument of high voltage sensitivity had therefore to be used. A triode of high grid insulation, with an anode potential less than the

ionization potential of the residual gas, was used [8]. The anode current for zero grid charge was compensated by a bridge circuit using another similar valve; the changes in it produced by charges on the grid were read on a galvanometer, calibrated by induction from a condenser; the charges could be read consistently to about $\frac{1}{2}$ per cent.*.

The source of light was a straight tungsten filament, the image of which, formed on the path of the slit, was 0·13 mm.

Fig. 2.



wide; the slit was of the same width. The speed of the pendulum when the slit passed the image was 84 cm./sec., so that the whole flash lasted about 1/3000 second, and the brighest part about 1/6000 second. This was of the

* We adopted a valve rather than, *e. g.*, a Compton electrometer in order to gain experience with it. We are still unconvinced that the valve electrometer has any real advantage for laboratory work. Its greater convenience, when it is in working order, is compensated by the constant attention it requires (chiefly owing to batteries running down) and the need for elaborate shielding from H. F. disturbances.

same order as the time-lag; the ideal conditions were therefore not fulfilled; but any great reduction in the time of the flash would have involved entire reconstruction of the apparatus, and it appeared that even this imperfect apparatus yielded some important information. In particular, it proved beyond a doubt that there is a time-lag and that the current continues to flow for an appreciable time after the light is turned off.

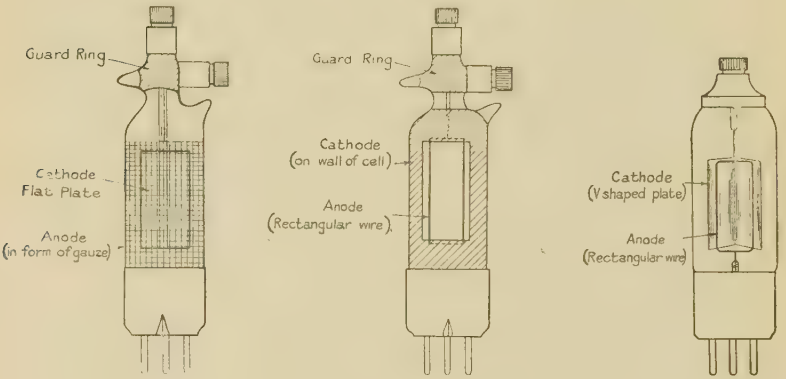
There was of course an appreciable interval between the opening of the contact by the pendulum and the opening of the switch A by the magnet; but no serious difficulty was found in making this interval constant. The setting of the micrometer screw at which the electrometer first received a charge was very constant; this setting was taken to correspond to $\tau = 0$. A much more serious difficulty, as was to be expected, lay in avoiding spurious charges. The whole electrical circuit, including batteries, was carefully shielded; it was found necessary to place the magnet at a distance of 50 cm. from the insulated electrode and to connect its armature to A by a glass fibre. The magnet was carried on a glass frame of the same material, so that thermal expansion did not alter the relative positions of the working parts. There was usually some "dark current" which caused the electrometer to receive a charge even when no light fell on the slit; but this could be readily allowed for by blank observations with no light.

6. An interesting fact was discovered in preliminary observations designed to test the apparatus. Since the mean current is independent of the frequency of interruption, q_m, the total charge passed by the cell in consequence of a single flash, should be $\bar{\imath}\mathrm{T}$, where $\bar{\imath}$ is the mean current passing during the flash and T its duration. q_m is the charge corresponding to $\tau = \infty$; since the variation of the light during a flash is always the same, $\bar{\imath}$ should be proportional to i_m, the current that flows when the slit and image of the filament are made to coincide, which can readily be measured. Accordingly q_m/i_m should be the same for all cells and all voltages on the cell; it should be a measure of T and determined wholly by the pendulum and the optical system.

Actually, however, in the test experiments q_m/i_m was found to vary in a ratio of at least 2 to 1, being least when the magnification was great, especially in cells of type 16 (see below). This anomaly was traced to the influence of charges on the walls of the cell, the area of bare glass being large in this type. It occurred only if the cell had been dark for

some seconds before the flash occurred and vanished when matters were arranged so that the light on the cell was interrupted only for a fraction of a second before the measured flash. For this purpose the pendulum was cut away so that light fell on the cell until the pendulum approached the position giving the flash. At this point the light was cut off, and immediately afterwards the electrometer, previously earthed, was insulated by switch B. With this arrangement q_m/i_m was found to be constant. It is clear therefore that light has to fall on the cell for a finite time before the current reaches a steady value ; during that time part of the current flows to the walls and not to the electrodes and the electrometer. After the steady state has been reached the

Fig. 3.



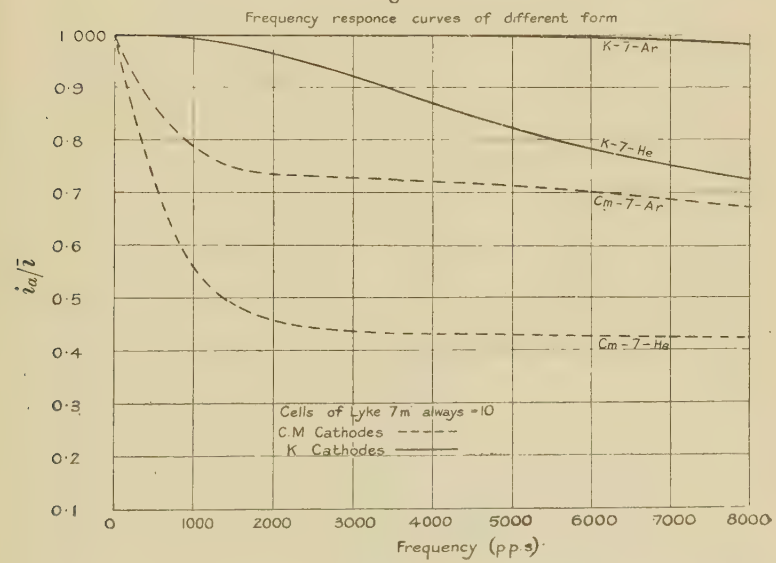
Type 6. Type 7. Type 16.

light may be cut off for a period not exceeding 0·25 second without the charges dissipating themselves materially by conduction. These charges on the walls seem to have been observed by Kingdon and Thomson [13], who did not, however, interpret their observations in the same manner.

These observations are important, because they tend to disprove any theory that would attribute the normal time-lag to charges on the walls. If a frequency-response curve is taken after the light has acted for a few seconds the effect observed here would not appear unless the period of interruption were as great as 0·25 second. It is not very probable that, if there is a time-lag of this order due to charges on the walls, there is another time-lag of very much shorter period, due also to the same cause, which is responsible for the decrease of $i_a/\bar{i}$ at frequencies of 1000 or more.

7. The cells examined were of the "Osram" types 6, 7, 16 made by the General Electric Company, Limited. Their construction is shown in fig. 3. It will be seen that in type 7 the cathode is on the walls, in types 6 and 16 in the centre of the bulb. The cathodes were either sensitized potassium (K) or cæsium on a mixture of Cs_2O and silver (CM), prepared by a process previously described [9]. The filling gas was either argon containing about 2 per cent. of nitrogen or helium very nearly pure. A symbol such as K – 7 – He means a cell of type 7 with a K cathode filled with helium. Unless

Fig. 4.



the contrary is stated the results given below refer to the cells used at the same *magnification*, namely $m = 10$.

III. *Results.*

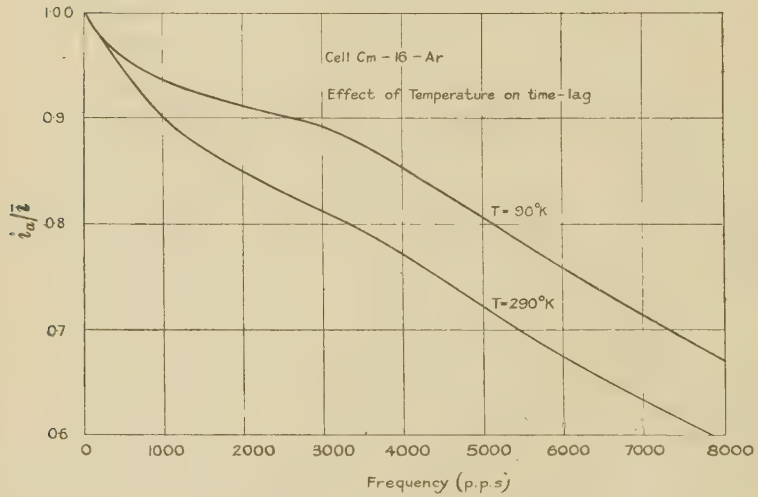
8. The best way to present our observations is to state the main conclusions to which they have led us:

(i.) *There are two kinds of time-lag, one effective at relatively low frequencies* (*L.F.*) *and one at relatively high frequencies* (*H.F.*).

This is suggested by the comparison of the curves for K – 7 – He and K – 7 – Ar in fig. 4 with those for CM – 7 – He and CM – 7 – Ar. It is possible that the difference between

the two kinds of curve may not be as great as is indicated in that figure. The CM curves *may*, like the K curves, start with a horizontal portion ; they *might* have a form generally similar to that of the K curves if they were stretched parallel to the axis of frequency. Observations at the low frequencies necessary to decide this point were not possible with our apparatus ; but even if this is so, the presence of two kinds of time-lag is indicated by the curve for CM—7—Ar and by the curve marked T = 90° K in fig. 5 ; for here the curve shows a relatively flat part at moderate frequencies, separating steeper portions at high and low frequencies. This

Fig. 5.



feature is also apparent in Metcalfe's curves [1], the earliest published, but it does not seem to have been recorded by later observers. No form for $f(\tau/\tau_0)$ at all similar to (6) could give rise to this feature ; if f is to be an exponential function it must have at least two terms with different time-lags, $\tau_1 = 1/\lambda_1$ and $\tau_2 = 1/\lambda_2$, with $\tau_1 < 10^{-4}$ sec. $< \tau_2$. It is not implied at present that these two time-lags are not theoretically connected, but only that they are experimentally separable.

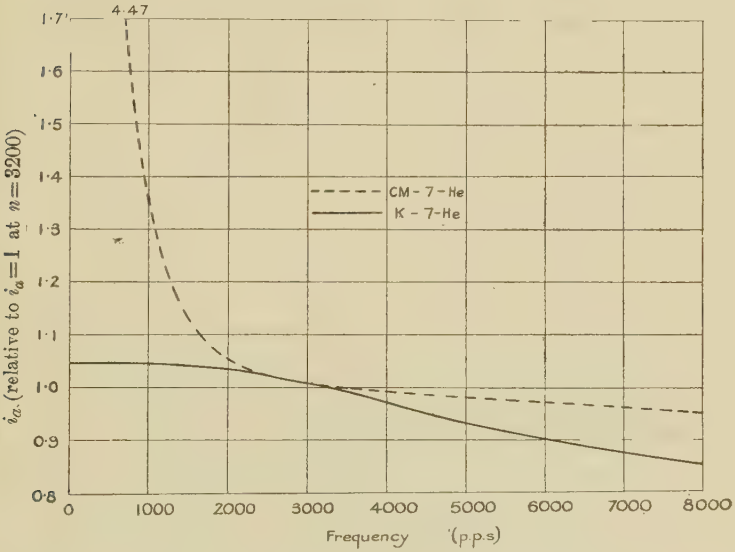
The H.F. time-lag characteristic, *e. g.* of K—7—He in fig. 4, could not be investigated by means of the pendulum, because, if it exists, it is concealed by the finite time of the flash. Cells showing frequency-response curves of this form

give pendulum curves indistinguishable from those of vacuum cells; but the L.F. time-lag can be investigated by the pendulum. Our next conclusion is based on pendulum curves.

(ii.) *In the L.F. time-lag τ_0 really increases with the magnification.*

The solid curves of fig. 2 refer to the same cell (CM—7—Ar) used at different magnifications m. That marked $m=1$ is obtained when the voltage is too low for ionization or when a vacuum cell is used; it merely demon-

Fig. 6.

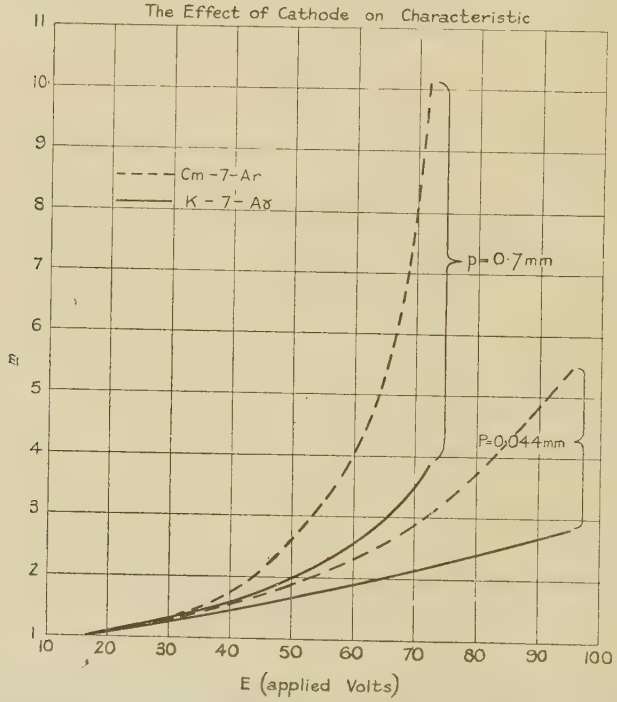


strates the finite duration of the flash. This duration makes it difficult—in theory impossible—to deduce unambiguously the form of f from the remaining curves. No detailed account of attempts in this direction need therefore be given; but it is quite certain that the assumption that f is always of the form (6), and that a but not λ varies with m, is inconsistent with these curves. In all the cells of every type there was clear evidence that the curves for different m really differed in time scale and that τ_0 (whatever the exact form of f) increases with m. The change of $i_a/\bar{\imath}$ with m is *not* due merely to an increase in the proportion of the current affected by time-lag.

(iii.) *The L.F. time-lag is connected with the cathode and probably with secondary emission from it.*

In cells with different cathodes but otherwise similar the L.F. time-lag is always much more marked in the cathode with low work-function, *e. g.*, CM as compared with K. The difference is shown in fig. 6, which shows the frequency-response curves of two cells, both of type 7, both filled with

Fig. 7.



helium to the same pressure, and both with the same applied voltage, but one with a CM and one with a K cathode. The mean currents through the cells are adjusted so that i_a is the same at a frequency of about 3000 ; at all other frequencies, and especially at low frequencies, i_a is greater for the CM cathode. The conclusion seems inevitable that there is a part of the current dominant in the CM, but almost absent in the K cell, that is subject to L.F. time-lag, while the part of the current subject to H.F. time-lag is relatively more prominent in the K cell.

Since the L.F. time-lag, though determined by the cathode, is absent when there is no ionization of the gas, it can hardly be attributed to anything but secondary emission at the cathode arising in some way from ionization of the gas. The presence of such secondary emission was neglected entirely in Townsend's original theory, and was probably completely absent in the experiments on which it was based, for they were conducted with cathode of large work-function. But the most superficial examination of the characteristics of modern gas-filled cells indicates its importance. Fig. 7 shows typical characteristics of cells of the same geometrical form, filled with the same gas to the same pressure, but with different cathodes. (Curves showing even more strikingly the same phenomenon have been given by Kingdon and Thomson[13]). Unless the most fundamental ideas of Townsend's theory are to be rejected the steeper rise of the cell with the CM cathode must be due either to greater secondary emission or to a difference in the gas. The second alternative is not inconceivable, for the presence of the CM cathode might possibly involve the presence of some impurity (*e. g.*, Cs vapour) in the gas, but it seems excluded by the experiments to be described immediately.

(iv.) *The L.F. time-lag arises in some process of thermal diffusion.*

This conclusion is based on the fact that, while the voltage-characteristic of a cell is almost or quite independent of its temperature, the frequency-response curve depends on the temperature. Fig. 8 shows how the curve for a CM—7—He cell is changed by reduction of its temperature from 290° to 90° K, while that of the KM—7—He is unchanged. Here the same voltage is used at both temperatures and corresponds to the same magnification. Fig. 5 shows the effect of temperature on a CM—16—Ar; here the fall in temperature reduces the magnification at a given voltage by about 10 per cent.; the voltage is higher at the lower temperature, so as to restore the magnification. In both cases the initial fall of the curve is steeper at the lower temperature, but the extent of the initial fall in CM—16—Ar is less. Cooling seems to affect chiefly the duration of the time-lag; it is difficult to see how temperature can affect a characteristic time unless that time is determined by thermal motion.

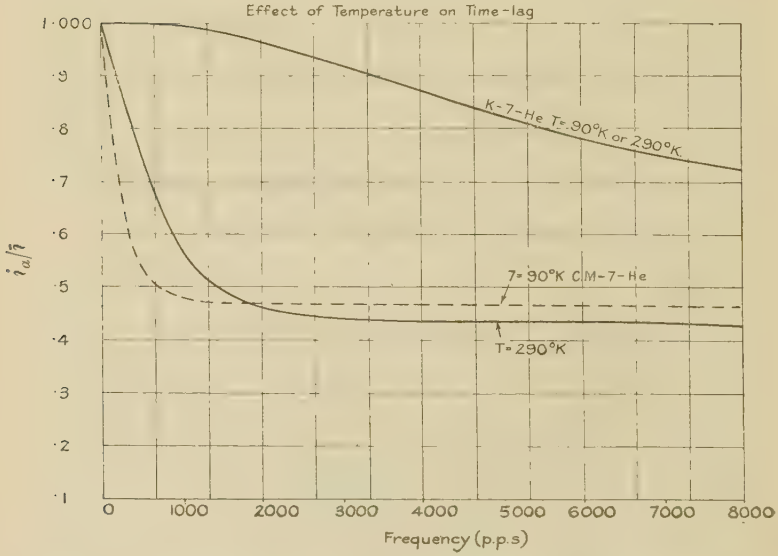
IV. *Theories.*

9. Conclusions (iii.) and (iv.) suggest a simple development of prevalent theories. We might suppose that

metastable atoms produced in the gas diffuse to the cathode and there produce secondary emission; time-lag would represent the period of diffusion. Many minor difficulties to this theory might be raised; thus it would be surprising if metastable He atoms, having an available energy of 19 volts, produced very different secondary emissions at cathodes both of which have work-functions less than 2 volts. But even more serious objections appear if we try to apply the theory quantitatively. Diffusion is governed by Fourier's equation

$$\frac{\partial N}{\partial t} = D\nabla^2 N, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Fig. 8.



where N is the concentration of the diffusion atoms and D their coefficient of diffusion. If metastable atoms behave like normal atoms

$$D = 1{\cdot}41\,\eta/\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where η is the viscosity and ρ the density [10]; or, if ρ is in mm. Hg and the temperature is 290° K,

$$D = 850/\rho \text{ for He}$$
$$= 95/\rho \text{ for Ar.}$$

To see how D will enter into $f(\tau/\tau_0)$ let us identify the cell with a sphere of radius r_0 with $N = N_0$ initially everywhere.

Then the secondary emission will be determined by the value of $-\frac{\partial n}{\partial r}$ at the boundary. The known[11] solution of (8) for this case is

$$-\frac{\partial n}{\partial r} = 2n_0(e^{-\mathrm{K}t} + e^{-4\mathrm{K}t} + e^{-9\mathrm{K}t} + \ldots), \quad . \quad (10)$$

where $\mathrm{K} = \frac{\mathrm{D}\pi^2}{{r_0}^2}$. If the right-hand side of (10) is to be proportional to $f'(\tau/\tau_0)$ we must clearly identify K in (10) with λ in (6) ; and this identification would not be upset if we took some other solution of (8); the factor r_0^2 alone would require change. Our cells are not actually spherical, but while we are concerned with orders of magnitude alone we may probably put $r_0 = 2$ cm. Then the extreme values of K for cells used in our experiments turn out to be given by $1/\mathrm{K} = 0{\cdot}14 \times 10^{-3}$ sec. for He at $p = 0{\cdot}3$, $\mathrm{T} = 290°$, and $1/\mathrm{K} = 3{\cdot}3 \times 10^{-3}$ sec. for Ar at $p = 0{\cdot}8$, $\mathrm{T} = 290°$. According to the theory, these times, differing in a ratio of more than 20 to 1, ought to be the time-lags τ_0 for these two extreme cases.

The pendulum curves for the extreme argon cell have been already given in the solid curves of fig. 2. The curve for the extreme He cell at $m = 10$ is given by the dotted curve. There is clearly no such difference in the time-lag for the He and Ar cells as the theory suggests. Moreover, any difference there may be between the two gases is much smaller than the differences arising from change of m, for which the theory suggests no simple explantion. Accordingly, though the theory of the diffusion of metastable atoms indicates time-lags of the right order of magnitude, we consider it is quite untenable.

10. We have no alternative theory to propound, for in our attempts to find one we always encounter a dilemma. If the time-lag arises in the performance of some motion and is determined by some velocity, this velocity must be derived either from thermal agitation or from the electric field ; it can hardly be derived from both because "thermal" velocities are so much smaller than "electric" velocities in the conditions relevant here. (Unless, of course, as in Langmuir's plasma, the energy of thermal agitation is derived from the field ; then it is almost independent of external temperature.) If the velocity is thermal the time-lag should depend on the temperature but not on the field, and if the velocity is electrical it should depend on the field

but not on the temperature. If, on the other hand, the time-lag occurs in the performance of some process that does not involve motions of atoms or ions, then it should not depend on either temperature or field. The inconvenient fact is that it seems to depend on both temperature and field.

These remarks apply at present only to the L.F. lag, for we have discovered nothing new about the H.F. lag, which alone Schröter and Lubszynski considered. But if it should turn out that here, too, the time-lag τ_0 (and not merely the fraction of the currents affected by time-lag) increases with the magnification, their theory will require modification.

Further, even if a theory could be devised to account for the facts presented here it would be impossible to test it properly in the present state of our knowledge ; for it would almost certainly involve some assumption about the processes that occur at the cathode and lead to secondary emission there, and we do not know what those processes are. It appears to us more profitable to return to the study of the Townsend discharge and to leave theories of time-lag until we know more of the processes occurring in it ; the investigation of time-lag should be one, but only one, of the means to such study.

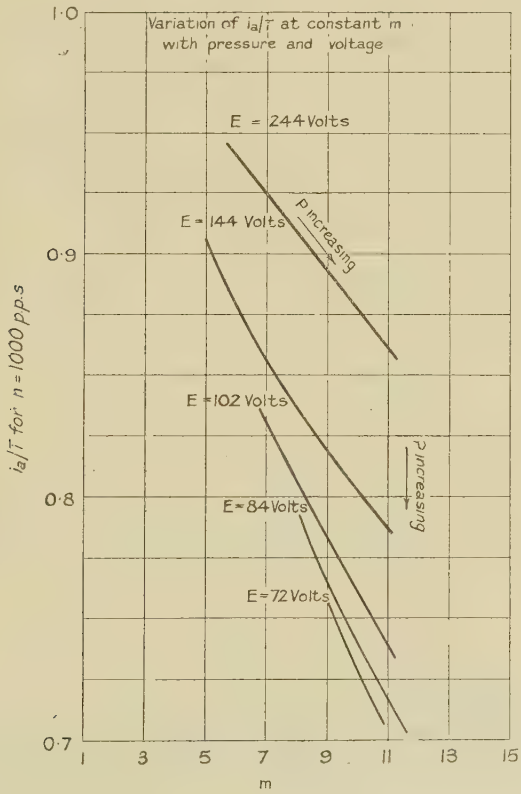
11. We would therefore record two tentative steps that we have made in this direction. The first arose in an attempt to answer the question, raised previously [3], on which basis cells filled with different gases or of different geometrical form should be compared in respect of time-lag. Practical utility suggests that these should be compared at the same magnification ; but has such comparison any theoretical basis ?

For this reason a CM—7—Ar cell was constructed in which the pressure of argon could be varied by cooling progressively a side-tube containing charcoal; characteristics and frequency-response curves were taken over a considerable range of pressures p and applied voltages E. The main conclusion was that $i_a/\bar{\imath}$ at a given n was *not* determined wholly by the magnification ; with different values of p and E giving the same m, $i_a/\bar{\imath}$ was not the same. $i_a/\bar{\imath}$ was less, and the fall in response with increasing frequency was greater, if p was high and E low than if p was low and E high, m being always the same. This is illustrated in fig. 9, where $i_a/\bar{\imath}$ at $n=1000$ p.p.s. is plotted against m for different values of E. Along each curve at constant E, p (not actually determined) varies, increasing with m ; at the same value of m, p is greater on the curve

corresponding to the smaller E. But no simple function of p and E could be found such that the same value of this function always corresponded to the same $i_a/\bar{\imath}$.

This conclusion naturally suggests—and, of course, it is otherwise probable—that even when the magnification is the same the proportions of the current carried by different

Fig. 9.



agencies (*e. g.*, primary electrons, secondary electrons, positive ions) may vary with the pressure and voltage, for some of these agencies are doubtless more affected by time-lag than others. It was hoped that some estimate of the proportion of secondary-emission current might be obtained by a comparison of characteristics, such as those of fig. 7, for geometrically similar cells filled with the same gas to the same pressure, but with different cathodes.

Suppose that each electron leaving the cathode produces α positive ions in passing through the gas, so that if there were no secondary emission and no recombination we should have $m=1+\alpha$. (If there is ionization by positive ions the ions so produced are still included in α.) In a given cell with given gas α varies with p and E but not with the nature of the cathode. Suppose further that the production of each positive ion is associated on the average with the emission of β secondary electrons from the cathode.

β will vary with the cathode; it will probably also vary with p and E, but there can be no certainty about this until we know how positive ions and secondary emission are related causally. But since changes of E and p, according to Townsend's theory, change the number rather than the nature of ionizing collisions, and since (see p. 468) there can be no considerable interference between independently emitted electrons, it is reasonable to assume provisionally that β does not vary very greatly with E and p. If so, the total number of electrons, primary and secondary, leaving the cathode in consequence of the emission of a single primary should be

$$1+\beta\alpha+\beta\alpha.\beta\alpha+\ldots\ldots=1/(1-\beta\alpha),$$

and m will be given by

$$m=(1+\alpha)/(1-\beta\alpha). \quad \ldots\ldots \quad (11)$$

If we compare two similar cells, distinguished by suffixes 1 and 2, identical in everything except the cathode, α should be the same for both, but β different. Accordingly we should have

$$\frac{m_1}{m_1-1}\cdot\frac{m_2-1}{m_2}=\frac{1+\beta_2}{1+\beta_1}=\mathrm{B}\ (\text{say}), \quad \ldots \quad (12)$$

where B should not vary very greatly with p or E.

Ideally this prediction should be capable of direct experimental test; actually there are certain difficulties. To estimate m we must know the primary emission which may be identified with the saturation current; actually there is no true saturation current in a vacuum cell—the current always increases somewhat with the voltage, especially in cells with "thin film" cathodes. Moreover, the impact of positive ions is known to alter somewhat the photoelectric properties of the cathode, and may therefore alter β; it is not really legitimate to assume that the cathode remains the same during the taking of a single characteristic. The uncertainties arising from these causes will not be discussed fully here;

we will merely give a table to show that there are conditions in which (12) is approximately true if plausible assumptions are made about the relation between primary emission and saturation current. The cells to which the table refer are a CM—7—Ar and K—7—Ar; the primary emission is estimated by an empirical rule that has been long adopted in routine measurements on commercial cells in these laboratories, viz., that it is 1·1 times the current through the cell when E=17, a value just too low to produce appreciable ionization.

p (mm.).	E (volts).	K cell, m_1.	CM cell, m_2.	B.
0·044	34·0	1·18	$1·21_5$	1·28
	47·5	1·44	1·62	1·25
	61·3	1·73	2·15	1·27
	68·0	1·88	$2·66_5$	1·26
	95·3	2·54	4·89	1·31
0·31	34·0	1·32	1·44	1·26
	47·5	$1·84_5$	2·35	1·26
	61·3	2·61	4·56	1·27
	68·0	3·13	8·26	1·29
0·83	34·0	1·23	1·30	1·24
	47·5	1·59	1·97	1·32
	61·3	2·24	2·50	1·29
	68·0	2·71	5·09	1·28
	71·5	3·00	7·34	1·30

The agreement with (12) is here remarkably good; B is much more nearly constant than either m_1/m_2 or $(m_1-1)/(m_2-1)$, the two factors of which it is composed. Unfortunately this agreement is exceptional. For a pair of cells of the same type filled with He, B increased regularly with voltage from 1·00 to 1·41, while in a pair of type 6 cells filled with argon it varied irregularly from 1·14 to 1·24. But there seems to be some evidence that the calculation is not entirely without significance and that B at the high values of m (where the known errors are least serious) does measure something characteristic of the difference between the two cells. If this is so the values obtained suggest that the difference between the secondary emission per positive ion from the CM and K cathodes is greatest in type 7 filled with

He and least in type 6 filled with Ar. (Of course B can give no information about the absolute value of β for either cathode, since it is possible, indeed probable, that β may be comparable with 1 even for the K cathode.) It so happens that this is also the order of the L.F. time-lags estimated from either the frequency-response curves or the pendulum curves. We are not disposed to insist on this difference ; we mention it merely as suggestive of the direction in which further investigation might be fruitful.

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(11) W. E. Byerly, Fourier Series (1895), art. 66.
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(13) K. H. Kingdon and H. E. Thomsom, ' Physics,' i. p. 343 (1931). (Unfortunately I did not see this paper until the experiments were completed and the paper written. If I had seen it earlier the research would certainly have taken a different direction at some points.)

LIII. *Some Properties of Porous Building Materials.*—VI. *An Autographic Apparatus for measuring Rates of Flow of Water.*

By E. Madgwick, *M.C., M.Sc., Ph.D., and* C. G. Webb, *B.Sc.**

Introduction.

THE absorption experiment described in Part IV. requires the continual presence of an observer. Some tests extend over such a period of time as to render this a serious inconvenience. Those on one-inch specimens of certain sandstones occupy eight hours or more, and in the case of

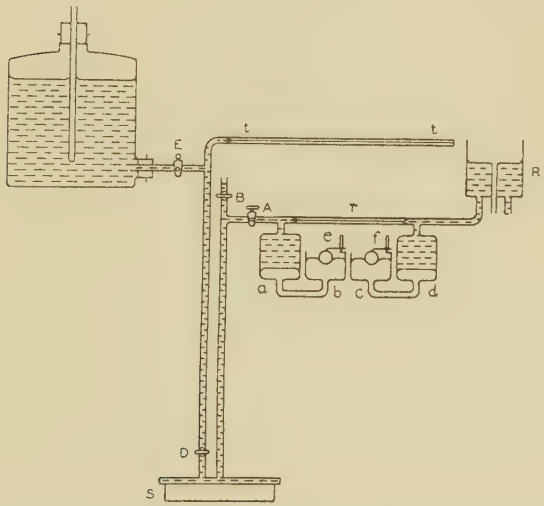
* Communicated by Sir F. E. Smith, K.C.B., C.B.E., F.R.S.

coated specimens the penetration may be much slower. Hence the need arises for an automatic method of recording rates of flow of water. By means of the arrangement to be described absorption curves of the type which have been discussed are registered directly.

The Apparatus.

The apparatus is shown in fig. 1. Water flows into the specimen S from a constant level reservoir R through a capillary tube *r* or other suitable resistance. The rate of flow is proportional to the fall of pressure across the

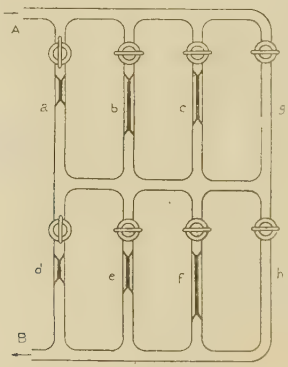
Fig. 1.



resistance, and this is measured by means of a manometer consisting of two similar U-tubes, *a b* and *c d*, containing mercury, the limbs *a* and *d* being connected to either side of the resistance. The limbs *b* and *c* are open, and each carries an optical lever, the two levers being identical. They consist of a light mirror supported by two steel points, which rest on a fixed platform, and by a steel ball, which floats on the mercury. The horizontal distance between the points and the centre of the ball is 1·5 cm., and the limbs of the manometer are all 4·6 cm. in diameter. The reflecting surfaces of the mirrors *e* and *f* face one another, so that a horizontal beam of light is reflected successively by *f* and *e* after it is brought to a focus on

a recording chart placed about 4 metres from *e*. The chart is of bromide paper wrapped on a drum which rotates by clockwork once in one, three, or twelve hours. This is enclosed in a light-tight case furnished with a vertical slit. The position of the spot of light corresponding to zero rate of flow is adjusted to be near the lower end of the slit. When the flow commences it is desirable that the deflected beam should strike the slit near the upper end, in which case practically the full length of the chart is utilized. The position of the chart being fixed, the sensitivity of the apparatus is varied by making the resistance *r* adjustable. Two forms of resistance have been employed, both designed for a maximum rate of flow of 0·4 c.c. per second.

Fig. 2.



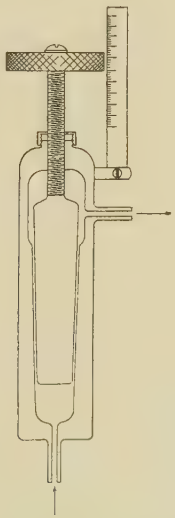
The first is shown in fig. 2. Water enters at A and leaves at B, having passed in the meantime through one of the capillary tubes *a*, *b*, and *c* or one of *d*, *e*, and *f*, or one of each set in series. By suitably choosing these tubes the resistance can be altered through a range of 20 to 1 in suitable steps.

The second, shown in fig. 3, has the advantage that it is continuously variable. The water flows along the annular space between two cones which can be moved relatively along their common axis. The solid cone is 8 cm. long and tapers from 11 mm. to 10 mm. in diameter. It can be raised through a distance of 40 mm. The pitch of the screw is 1 mm., and the micrometer head is divided into hundredths. The resistance is calibrated directly in cm. deflexion for unit flow at 1 metre scale distance.

This apparatus was designed to yield absorption curves without correction in the case of normal test specimens of 10 inches diameter, in accordance with the following considerations :—

(i.) Since the pressure drop in the adjustable resistance is varying during the experiment, its maximum value must be so small as not to affect appreciably the total head on the specimen. The capillary head is usually considerably greater than 100 cm. Using an applied head of 100 cm. a pressure drop up to 2 cm. of water is permissible at the beginning of a test.

Fig. 3.



(ii.) The angular deflexion of the beam of light should be so small that the linear deflexion of the image is proportional to the rate of flow. This is true to the accuracy required in our experiments for deflexions up to 6°.

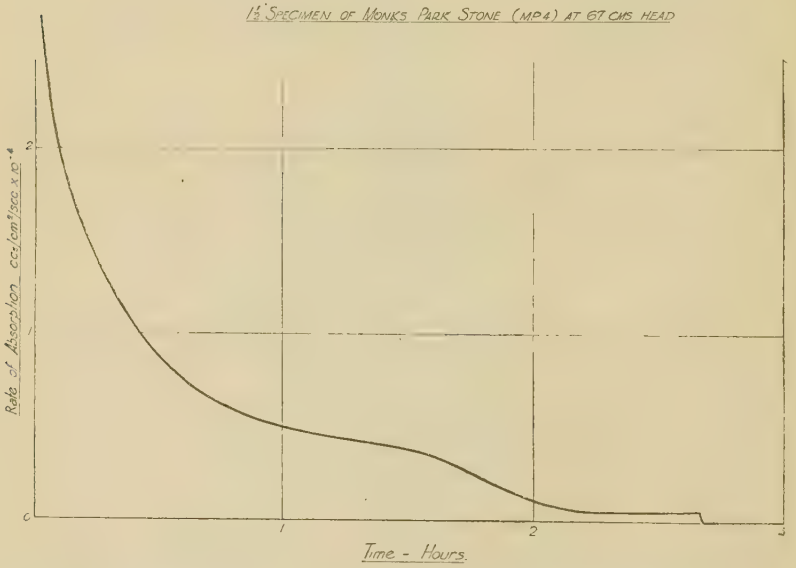
(iii.) The diameter of the manometer tube needs to be great enough to eliminate surface tension effects, but sufficiently small that the volume of water displaced from the manometer shall be negligible in comparison with the total absorbed by the specimen.

Method of Experiment.

The tap A (fig. 1) being closed, the specimen is connected to the apparatus. The space above the specimen and the

connecting tubes are filled with water by opening the taps B, D, and E, a flash of light recording on the chart the moment the specimen is wetted. When all the air has been driven out through B these three taps are closed and A is opened. The flow then proceeds into the specimen from R through r. The beam of light is immediately deflected, and the resistance r is adjusted until the deflexion is of a suitable amount. As absorption proceeds the spot of light moves slowly down the chart, tracing out the absorption curve, until it arives at a constant level corresponding to the steady

Fig. 4.



rate of transmission. To ascertain when this stage has been reached the deflexion is noted occasionally, a scale being provided near the slit for that purpose. The flow is then stopped by closing A, and the deflected beam returns to its zero position. While this is being recorded it is customary to measure the rate of transmission directly by the method described in Part IV., a capillary tube t t being permanently fitted to the apparatus. This has the double advantage of serving as a check on the calibration and of obtaining an accurate value of the rate of transmission when it is small. If the capillary tube is not employed the deflexion corresponding to steady transmission may be increased at the end of an experiment by again adjusting r.

When necessary, a correction for the displacement of water from the manometer may be applied as follows :—If p is the pressure drop in the adjustable resistance, the rate of flow through the resistance is proportional to p, and is represented by the displacement Q′ on the chart. The rate of withdrawal of water from the manometer is proportional to $\frac{da}{dt}$, and therefore to $\frac{dQ'}{dt}$. The flow of water into the specimen is thus

$$Q = Q' + c\frac{dQ'}{at},$$

where c is a constant which can readily be evaluated.

Fig. 4 shows a typical absorption curve which has been obtained by this method,

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LIV. *Some Properties of Porous Building Materials.*—VII. *The Problem of Damp Walls.* *By* E. Madgwick, *M.C., M.Sc., Ph.D.**

Introduction.

DAMPNESS in walls, resulting in a lowering of the thermal resistance, discoloration, and efflorescence, often attended by disintegration, may be due to the flow of water from the outside to the inside or upwards from wet foundations. These factors may be examined to a limited extent on the principles which have been developed.

Water may also be condensed on the inside surface and, if allowed to collect, will have a deleterious effect on pictures and other articles of furniture. Experiments on the moisture condensation on various surfaces from a humid atmosphere have been carried out under the auspices of the Building Research Board [11]. One face of each specimen was exposed to steam and the other to cold air. The surfaces were observed at intervals and arranged roughly in order in respect of the quantity of water visibly deposited. It is of greater practical importance to know whether under specified conditions such a visible deposit will or will not be formed, and the problem will be examined here from that aspect.

The Flow of Water through Walls.

The horizontal and upward flow of water in walls occur under conditions which vary enormously and which in

* Communicated by Sir F. E. Smith, K.C.B., C.B.E., F.R.S.

general are not sufficiently simple to admit of calculation. Walls are not homogeneous; the outsides are generally wetted intermittently, and the supply of water at the surfaces or foundations may be insufficient to provide for the maximum rate of absorption which is possible. Nevertheless calculations under assumed extreme conditions are frequently helpful, as the following example will illustrate.

Consider a wall of thickness $l=23$ cm. (about 9 inches) composed of a certain sand-faced brick whose absorption constants were found to be:

Capillary head	$h'=95.$
Resistivity	$\sigma=0{\cdot}336\times10^8.$
Penetration constant ...	$D=24.$
Porosity	$B=0{\cdot}130.$

Suppose the outside to be maintained in a wet condition, and, further, that there is a wind pressure equal to that due to a head h of 15 cm. of water. Then, by equation (24), the water will penetrate to the inner surface of the wall in time

$$t=\frac{Dh'l^2}{h'+h}\text{ seconds},$$

or about 3 hours. Afterwards it will be driven through the wall by wind pressure at a rate

$$q=\frac{hg}{\sigma l}\text{ c.c. per sq. cm. per second},$$

or 685 c.c. per sq. metre per hour. It is clear that, even under conditions which are not nearly so extreme, a wall of such a nature would be most unsatisfactory.

Next suppose a thickness $l_1=1$ cm. of a cement rendering to be applied to the outside. As we have seen in Part V. the resistivity of such a rendering alters when it absorbs water, but a working minimum value may be determined by means of the absorption apparatus. Adopting the moderate value for a good rendering of $\sigma_1=5\times10^{10}$ C.G.S. units, the minimum time in which water can flow from the outside to the inside of the brick portion of the wall is, by equation (30),

$$t=\frac{Bl(\sigma_1 l_1+\frac{1}{2}\sigma l)}{(h'+h)g}\text{ seconds},$$

or about sixteen days. The defect is thus adequately corrected.

The Condensation of Water on Wall Surfaces.

When the temperature T_1 of the air inside a room is greater than that of the air outside there is a drop from T_1 to T_2, the temperature of the inside surface of the wall, across a layer of air of the order of 1 cm. in width. If T_2 is below the dew-point water will be condensed. This water may remain on the surface, the small drops collecting and running down the wall, or it may be absorbed so rapidly that it is never visible, in which case the wall acts as a sort of reservoir, storing the water until condensation has ceased and later giving it up by evaporation. The collection of water on the surface of a wall may thus be prevented by the provision of a lining which is able to absorb water at a rate at least equal to that at which it is condensed, and of sufficient capacity to hold all that may be deposited under practical conditions.

In order to make allowance for the extreme case we shall assume the air in the room to be saturated, and shall consider the relationship between the rate of deposition and the rate of transfer of heat from the air to the wall. Let m be the mass of water vapour necessary to saturate 1 gm. of air. Then the heat liberated by a quantity $M+mM$ grams of saturated air in cooling at constant pressure through a temperature δT is

$$\delta H = M\left(S_1 + mS_2 + L\frac{dm}{dT}\right)\delta T,$$

neglecting that due to the cooling of the condensed water, which, for the range of temperatures with which we are concerned, is small ; where

S_1 = specific heat of air at constant pressure,

S_2 = specific heat of water vapour,

L = latent heat of water vapour

at the appropriate temperature. Hence the amount of water condensed is

$$\delta m = M.\frac{dm}{dT}.\delta T = C.\delta H, \quad . \quad . \quad . \quad . \quad (71)$$

writing

$$\frac{dm}{dT}\Big/\left(S_1 + mS_2 + L\frac{dm}{dT}\right) = C.$$

The values of C are near the round numbers 9×10^{-4} at 10° C., 10×10^{-4} at 15° C., and 11×10^{-4} at 20° C. The extent of the cooling under the conditions we are considering

is from zero up to $T_1 - T_2$, which is of the order of only a few degrees. It is a close enough approximation to treat C as constant and corresponding to the mean temperature $\frac{1}{2}(T_1 + T_2)$. We then have the simple result that the rate of deposition of water is practically proportional to the rate of transfer of heat, which can be estimated from the overall coefficient of the wall and the inside and outside air temperatures.

To test this relationship experiments have been carried out on the deposition of water on plaster specimens by means of the condensation apparatus to be described. Thermocouples were embedded in the plaster at known distances d apart, and the steady temperatures t_1 and t_2 which they attained were measured, the specimen being weighed before and after an exposure lasting some hours. By equation (71) the theoretical rate of deposition is

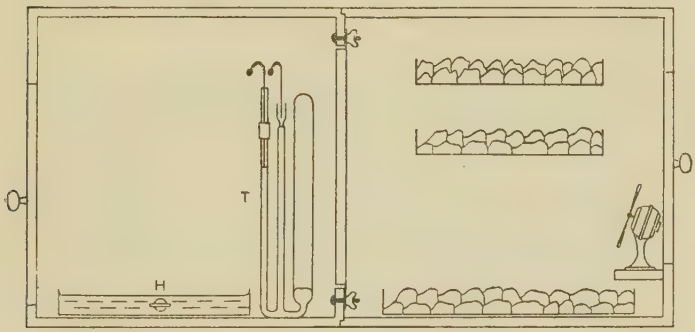
$$\frac{CK(t_1 - t_2)}{d} \text{ c.c. per sq. cm. per sec.,}$$

where K is the thermal conductivity of the plaster. This has the value 0·0011 C.G.S. units when the plaster is dry, but is considerably higher when it is wet. There appear to be no figures giving the variation of this quantity with moisture content, but Griffiths [12] has recorded that the conductivity of a wet wall fell from ·0054 to ·0021 in drying out. The plaster specimens in the present experiments were generally bordering on saturation towards the end of the tests, so that the average value of the thermal conductivity during each test must be considerably greater than that of the dry specimen, which, in the absence of further data, has been employed in the calculations. Nine experiments were performed under different experimental conditions, the theoretical rates of deposition ranging from $2{\cdot}52 \times 10^{-6}$ to $13{\cdot}7 \times 10^{-6}$ c.c. per sq. cm. per second. The observed rates were always higher, the mean ratio being 2·3 to 1, which is of the order to be expected owing to the increase in conductivity due to the absorption of water. The expression (71) thus appears to give at least a rough quantitative explanation of the facts, and has been adopted provisionally.

Applying it to the case of a wall of overall coefficient 5×10^{-5} calories per sq. cm. per sec. per degree Centigrade difference in temperature, with inside and outside air temperatures 20° C. and −5° C. respectively, and taking the value of C corresponding to 17° C., we have for the rate of deposition $1{\cdot}3 \times 10^{-6}$ c.c. per sq. cm. per sec. Materials which fail to absorb water at this rate when tested under

zero applied pressure would be expected to collect water on their surfaces under the conditions specified. Such include painted or sized plasters. All untreated natural stones, plasters, and bricks are able to absorb water at a rate greatly in excess of this, and are immune from the effect. These predictions may be applied with confidence. In addition there are border-line and doubtful cases—an absorption test on a coated specimen, for example, may be affected by imperfections in the coating—which are further examined by means of the condensation apparatus.

The estimation of the capacity of a wall-lining for holding water presents no difficulty. Moreover, this is a factor which need rarely be considered, as will be appreciated from the fact that under the extreme conditions which have been discussed the quantity of water deposited in an hour is only 0·0047 c.c per sq. cm.



The Condensation Apparatus.

The condensation apparatus is shown in the figure. The specimen forms a partition 11 inches square between two chambers. One chamber contains three trays of ice, the air being kept circulating by means of a fan. The air in the other is maintained in a warm and humid condition, the temperature being adjusted by means of a thermo-regulator T which operates an immersion heater H contained in a tray of water. Provided the heating is not too drastic the temperatures of both air and water remain reasonably constant, and a high degree of humidity is ensured by the fact that the temperature of the water is greater than that of the air. Each side of the chamber is provided with a window for viewing the specimen, and an inspection lamp is placed in a suitable position. In making a test it is necessary to decide

on the heat-flow for which it is desired to allow. This is calculated for specified inside and outside temperatures from the overall coefficient of the wall on which the material is to be used. Thermo-couples are fixed to the faces of the specimen and the thermo-regulator is adjusted until they indicate the temperature difference corresponding to this heat-flow. Specimens of unknown thermal conductivity, or thin sheets in which it would be difficult to measure the temperature gradient, can be fixed to standard slabs of known thermal conductivity. The specimens are left on test for times which are determined by the total quantity of water they are required to absorb.

It has been found, in accordance with the considerations outlined above, that condensed water does not appear visibly on bare plaster even under the most rigorous conditions, but collects readily on the surface when it is painted or sized. A bare deal board, which shows no effect when tested under ordinary dwelling conditions, is unsatisfactory when painted or oiled.

The writer is indebted to the Department of Scientific and Industrial Research for permission to publish an account of this investigation.

References.

(11) Building Research Board. Special Report No. 4. H.M. Stationery Office (1921).
(12) Building Research Board. Special Report No. 7 (1923).

LV. *Formation of Rust which shows the Configuration of Stream-lines.*

By Fukusaburo Numachi, *Dr.(Japan), Assist. Prof of Hydraulics and Hydrodynamics, Tohoku Imperial University, Sendai, Japan* *.

[Plates VII.–IX.]

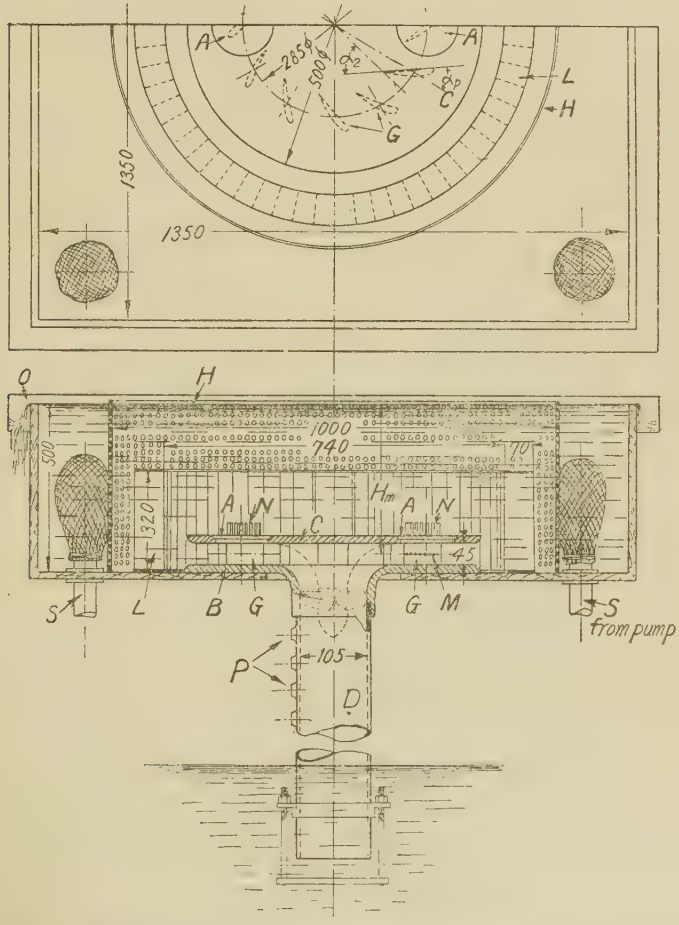
SINCE photographs of the stream-lines of fluids were found to play an important rôle in the research of hydrodynamics, especially of modern aerodynamics, various methods have been considered in connexion with photographs, such as the aluminium powder method, the smoke method, and the dyestuff or colour method.

The writer undertakes in this thesis to report on his discovery of a phenomenon concerning the formation of rust

* Communicated by the Author.

which shows the exact configuration of the stream-lines caused by flowing water. This phenomenon, he believes, is not only very useful in the photographic study of stream-lines, but will also be of much interest from the physical and chemical standpoints.

Fig. 1.



The noteworthy fact is that the phenomenon revealed here by the photographs can be applied to such a case as a flow along a complicated boundary such as a helical surface or a flow along a boundary which cannot be made transparent. For these cases the methods hitherto known, some of which are mentioned above, can never be adopted.

In the course of the experiment concerning the flow of water through guide-vanes with an aerofoil profile arranged in a grate-form with a circular grate-axis, as is shown in figs. 1 and 2–4 (Pls. VII.–IX.), the writer happened to find some rust formed on a plate of cast iron, by which the figure of the stream-lines was exactly and distinctly shown.

Then, in order to make use of the figure, he modified the process of the formation of the rust until finally he succeeded in easily taking clear photographs.

Originally the apparatus was set up as follows* :—The guide-vanes of an aerofoil profile, G in fig. 1, were arranged between two circular plates B and C, and the water was supplied by a pump through four tubes, S, opening respectively near the four corners of a square tank, in which the surface of the water was maintained constant. The water, first passing through honeycomb, H, and director, L, was made to enter the aerofoil-grate, G, then collected in the suction-tube, D, arranged with a series of holes, P, for inserting a Pitot-tube.

All the aerofoil profiles were made of brass, 4·5 cm. in height and 10 cm. in chord length, and the two circular plates, B and C, were made of cast iron. The series of holes, M, on the surface of the aerofoils, to which the tubes, N, respectively communicated, and the two disks, A, fitted into the upper plate, C, all of which belonged to only two aerofoils set on the opposite sides of the grate, were merely the device for measuring the distribution of the pressure on the surface of the profiles.

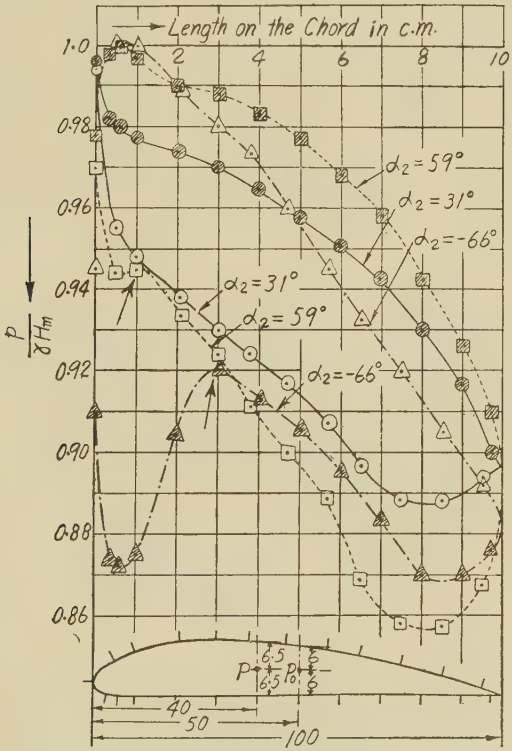
The upper side of the lower plate, B, and the under side of the upper plate, C, were covered with the figures of the stream-lines made by the rust, some of the photographs of the latter figure being shown in figs. 2–4 (Pls. VII.–IX.).

The method of improving the production of the figure was considered in several ways Firstly, a powder of the rust of cast iron was poured into the water, by which the contrast of the figure was strengthened, as is shown in fig. 2 (Pl. VII.). Secondly, small holes of 0·5 mm. diameter and 1·0 mm. depth were drilled, into which vaseline was preliminarily

* With this apparatus the author intended to perform the experimental study of propeller turbines and propeller pumps, since he had contributed some papers on the theoretical study of the machines, which were published in Technology Reports of Tohoku Imperial University, Sendai, Japan, vol. iii. no. 3, and vol. iv. no. 2, as well as in the Report of World Engineering Congress, Tokyo, Japan, 1929, Paper No. 71, entitled "Aerofoil Theory of Propeller Turbines and Propeller Pumps, with Special Reference to the Blade Interference upon the Lift and the Cavitation."

put ; when the water was made to flow the vaseline was induced out of the holes and pushed forward gradually in the passage of a stream-line, where the metal remained unrusted and brilliant, as may be seen especially in fig. 3 (Pl. VIII.). In fig. 4 (Pl. IX.) the case is shown when the quantity of vaseline was rather small.

Fig. 5.



α_p	α_2	v_t m/sec.	H_m c.m.
17·2°	31°	0·388	45·81
38·7°	59°	0·308	45·76
42·9°	-66°	0·349	45·85

The time required for the plate to be covered completely by the rust was about from ten to thirty minutes, depending upon the temperature. The temperature at which the experiment was carried out was from 6° to 20° C., the mean velocity v_t of flow at the leading edge of the aerofoil profile being written in the figure.

These figures have given the writer much aid in deducing the conclusion * of the experimental result by contrasting them with the pressure distribution on the surface of the profiles and the condition of the flow in the suction-tube S.

For instance, the curve of pressure distribution on the aerofoil profile obtained has shown that in the case when angle of incidence α_2 is 59° or −66°, as in figs. 2 and 3 (Pls. VII. & VIII.) respectively, the pressure rises again at the place near the leading edge of the back surface, as is shown in fig. 5. In fig. 5, p/γ in the ordinate is the surface pressure in meter of water column, H_m being shown in fig. 1. The reason therefore being, as shown clearly by the very figures, that the water, which stagnates at the front surface of the profiles, flows around the tip of the profiles and impinges on the place now under consideration. For another instance, the trailing vortex, which is produced by the difference of the velocities on the pressure and vacuum sides of the aerofoil profile, is clearly observed as to its passage and as to its shape, as is seen in figs. 2–4 (Pls. VII.–IX.).

The merit of the application of this phenomenon is as mentioned above, the only demerit being that the figure of the stream-line is limited to one in contact with the boundary.

This method can be applied to the flow of air and of other gases as well by adopting an adequately erosive gas or corrosive metal as the boundary.

The writer wishes to take this opportunity to express his gratitude to Prof. O. Miyagi for his encouragement, and his thanks also to Mr. K. Tsunoda and Mr. I. Mashiko for their help in connexion with this experiment.

Berlin.
March 28th, 1932.

LVI. *The Effect of an Axial Wire on the Flow of Water through a Circular Pipe.* *By* A. Fage, *A.R.C.Sc.*, *and* H. C. H. Townend, *B.Sc.*†

Summary.

In some recent experiments by Prof. Lea and Dr. Tadros measurements were made of the effect of an axial wire on the rate of discharge of water from a circular pipe at low

* A full report will be published in Technology Reports of Tohoku Imperial University, Sendai, Japan.

† Communicated by the Authors.

Reynolds numbers. The reduction of flow due to a wire was found to be considerably smaller than that predicted by the theory of stream-line flow. It was suggested that this difference was due to a turbulent core in the immediate neighbourhood of the wire.

To ascertain whether such a turbulent core exists, the flow near an axial wire has been viewed with an ultra-microscope. The turbulence present was hardly perceptible, and accordingly measurements of the ratio of flow with and without an axial wire were made. These measurements are at variance with those made by Prof. Lea and Dr. Tadros. The rate of discharge from a pipe with an axial wire was found to be in close agreement with the theoretical rate, whilst the appearance of the flow under the ultra-microscope did not suggest a concentration of turbulence near the wire.

Introduction.

(1) AN investigation by Prof. Lea and Dr. Tadros of the effect of an axial wire on the flow of water through a circular pipe at low Reynolds numbers is given in Phil. Mag. no. 74 (June 1931). At the beginning of the paper the theory of stream-line flow is used to obtain an expression for the ratio of the rate of discharge with an axial core to that without a core for the same pressure gradient down the pipe. The expression * obtained was

$$\frac{q}{Q}=\left[1-\left(\frac{d}{D}\right)^4-\frac{\left[1-\left(\frac{d}{D}\right)^2\right]^2}{\log_e\frac{D}{d}}\right],$$

where d and D are the diameters of the core and pipe respectively and q and Q the rates of discharge with and without the core respectively.

Values of q/Q calculated from this expression for various values of D/d are given below :—

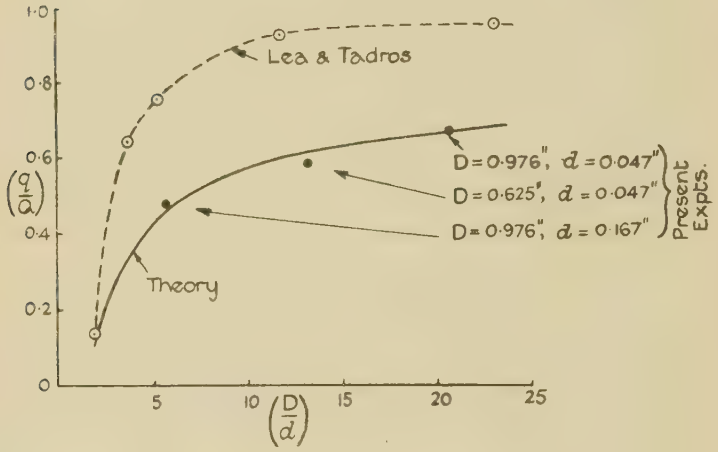
D/d.....	2	4	5	10	20	50	100	1000
q/Q.....	0 26	0·362	0·426	0·574	0·668	0·745	0·780	0·855

Commenting on these values, the authors say that it appeared difficult to imagine that an axial wire of diameter 1/1000th of that of the pipe could diminish the rate of

* Notation changed.

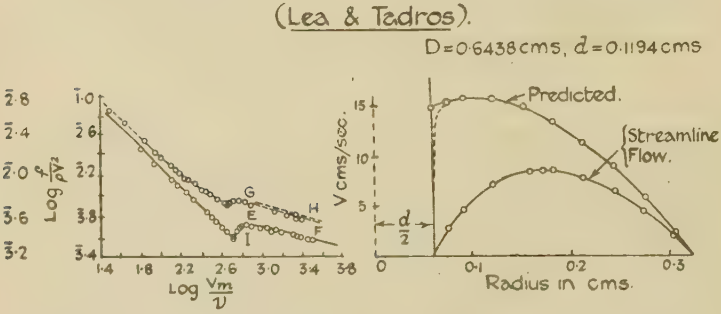
discharge by 14·5 per cent., and so they considered it was desirable to measure the effect of a wire. The values of q/Q obtained are compared with the theoretical values in fig. 1 of this paper. It is seen that the tests indicated that when D/d had a value of 2 the theoretical and experimental

Fig. 1.



Fig, 2.

Fig. 3.



discharges were in close agreement, but when a wire of relatively small diameter was used the effect of the wire was much smaller than that predicted by theory.

(2) Some of the experimental curves obtained are reproduced in fig. 2, where values of $\log(f/\rho V^2)$ are plotted against $\log(Vm/\nu)$, where m is the hydraulic mean depth, V is the mean speed of flow, f is the intensity of surface

friction, and ρ and ν are the density and kinematic coefficient of viscosity of the water.

The two curves are displaced vertically for clearness. It is seen that for the circular pipe alone the point, I, at which instability occurs is well defined, but for the pipe with an axial wire some departure from stream-line flow clearly occurs earlier. Prof. Lea has informed us that in a similar pipe to that in which the wires were placed, and taking special precautions, he could obtain a value of VD/ν for the point I, without a wire, of well over 9000 quite easily.

(3) Possible explanations of the differences between theory and experiment are advanced in the paper, and the conclusion arrived at by the authors is that "the difference . . . . . appears to necessitate either definite slip at the boundary of the core or else a much more rapid rate of change of velocity at this boundary than would be the case if there were no * slip." They say that the difference can be accounted for by assuming that near the surface of the wire there is a thin film in which an early breakdown of flow occurs, and that this early disturbance causes the turbulence to be confined to the region close to the wire.

On these assumptions a prediction is made of the velocity distribution required to produce the rate of discharge measured for a pipe of diameter 0·6438 cm. with a wire of diameter 0·1194 cm. It is seen in fig. 3 that this predicted distribution differs considerably from the distribution for stream-line flow.

Present Experiments †.

(4) Recently an ultra-microscope has been used by us to examine turbulent flow in pipes ‡. In this work it was found that minute particles in ordinary tap-water became visible when intensely illuminated, and that it was possible to follow their motions when they were viewed through a microscope against a dark background. The experience gained from this work led us to believe that if a turbulent core exists at the surface of an axial wire it should be possible to detect its presence with an ultra-microscope.

* The sense of the quotation suggests that this should read ". . . . . if there were slip."

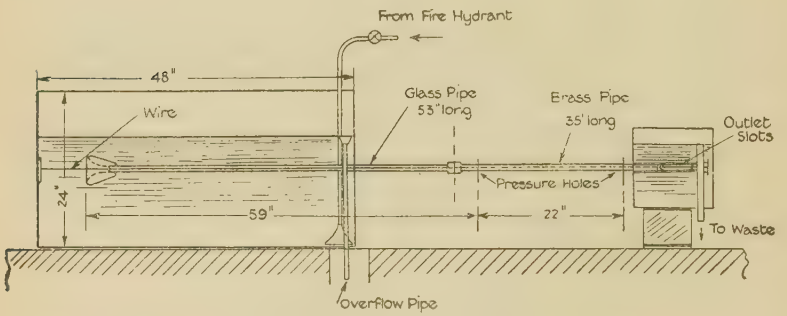
† These experiments were made in the Aerodynamics Department of the National Physical Laboratory, and permission to communicate the results was kindly granted by the Aeronautical Research Committee.

‡ "An Examination of Turbulent Flow with an Ultra-microscope," Fage and Townend, Proc. Roy. Soc. **A**, cxxxv. (1932).

Accordingly the flow near a wire of diameter 0·167 in. mounted in a square pipe of side 0·89 in. was observed in this manner. The impression formed from a somewhat cursory examination at speeds just below the critical value was that the flow was apparently rectilinear. Further, it was difficult to believe that any deviation from rectilinear motion small enough to escape detection could account for the large differences between the theoretical and experimental rates of discharge obtained by Prof. Lea and Dr. Tadros. There appeared to be no alternative but to conduct some experiments similar to those made by Prof. Lea and Dr. Tadros before further examination with the microscope.

(5) The principal measurements were made with an aluminium wire of diameter 0·047 in. mounted in tension

Fig. 4.



along the axis of a $\frac{5}{8}$-in. pipe. A sketch of the apparatus is given in fig. 4. The pipe had a total length of 87 in. and was built up of 53 in. of glass pipe followed by 34 in. of brass pipe. The junction between the two pipes was made with some care. The glass pipe was used to allow the flow to be observed and also to permit correct axial alignment of the wire. To obtain steady conditions of flow the entry of the pipe was carefully faired with a converging bell-mouth, and the water-supply was obtained from a large tank (4 ft. × 2 ft. × 2 ft.). The pressure drop over a length of 22 in. of the brass pipe was measured on a 13-in. Chattock gauge in which water was used in contact with carbon tetrachloride of specific gravity 1·605 (15·2° C.). The first pressure hole was at a distance of 59 in. (94 diameters) from the mouth of the pipe. The mean rate of flow, $\bar{V}$, was obtained from a direct measurement of the rate of discharge.

The head was varied by adjustment of the heights of the overflow pipes in the supply- and discharge-tanks.

(6) The results obtained are given in fig. 5, where values of $\log_{10}(f/\rho V^2)$ are plotted against $\log_{10}(Vm/\nu)$. The values of f were calculated from the relation $f = m.dp/dl$, where dp/dl is the pressure gradient down the pipe. With the wire in place the symbol f represents the sum of the frictional forces on the surfaces of the pipe and wire divided by their combined area.

It will be seen that the results for the pipe alone lie very closely on the theoretical curve

$$\frac{f}{\rho V^2} = 2\left(\frac{\nu}{Vm}\right),$$

and that the departure from stream-line flow occurs near the value of Vm/ν obtained in the well-known experiments of Stanton and Pannell. Except near the critical point, the observations for the pipe with the axial wire lie closely on a straight line

$$\frac{f}{\rho V^2} = 2{\cdot}91\left(\frac{\nu}{Vm}\right).$$

The constant in this relation differs from the theoretical constant 2·745 by 6 per cent. It will be observed that a departure from stream-line flow begins at $\log_{10}(Vm/\nu) = 2{\cdot}255$, $(Vm/\nu) = 180$). This departure is, however, much more gradual than that observed by Prof. Lea and Dr. Tadros for values of D/d of about the same magnitude.

(7) To calculate the value of q/Q from those parts of the curves of fig. 5 which refer to the stream-line flow use was made of the relation

$$q = \frac{\pi}{4}(D^2 - d^2)V = \frac{\pi}{4\mu}(D^2 - d^2)m^2 . \frac{dp}{dl} . \left(\frac{\mu V}{fm}\right).$$

It follows from this relation that the ratio of the rate of discharge with the wire to that without the wire at the same value of dp/dl is given by

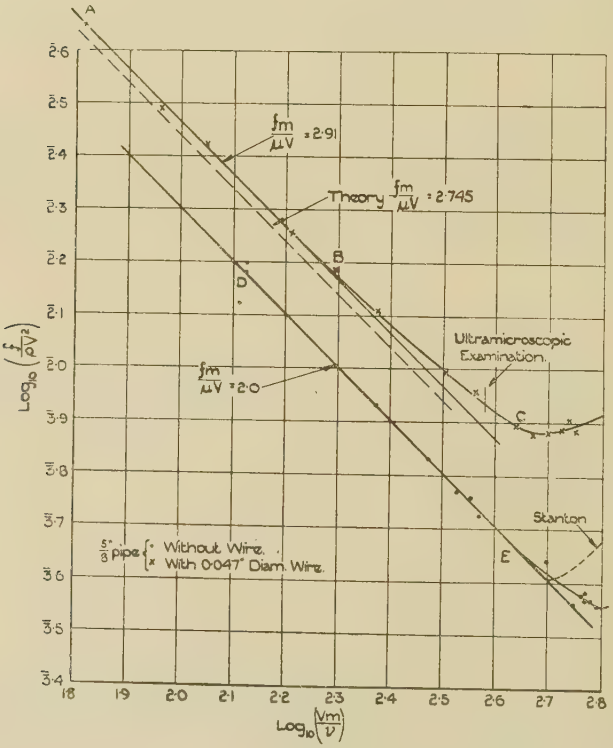
$$\frac{q}{Q} = \left(1 - \frac{d^2}{D^2}\right)\left\{m^2\left(\frac{\mu V}{fm}\right)\right\}_{\text{with wire}} \Big/ \left\{m^2\left(\frac{\mu V}{fm}\right)\right\}_{\text{pipe alone.}}$$

The experimental values of $(fm/\mu V)$ are 2·91 for the pipe with wire and 2·0 for the pipe alone. The corresponding value of q/Q is 0·583.

(8) The value of q/Q can also be obtained from the curves of fig. 6, where values of dp/dl are plotted against q and Q.

This figure shows that the above value of q/Q is legitimate for values of $(dp/dl) < 0{\cdot}10$, since the flows both with and without the core are then stream-line. The letters in fig. 6 correspond with those in fig. 5, where it can be seen that the range $dp/dl < 0{\cdot}10$ involves a comparison of the part AB of the curve for the pipe with wire with the part DE of the curve for the pipe alone. It is not permissible to compare

Fig. 5.

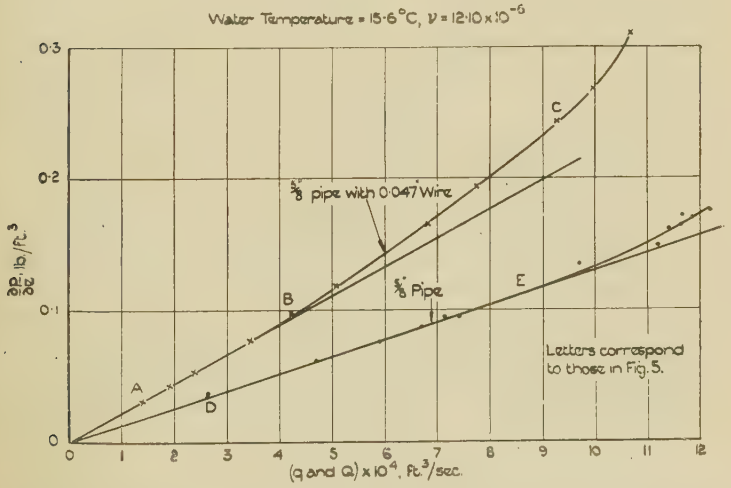


the part BC with the corresponding part for the pipe alone, since at the same value of dp/dl the flow in the pipe alone has become turbulent. Hence this part of the curve cannot be dealt with by calculations of q/Q obtained at constant dp/dl. It will be recalled that it was a difference between the experimental and theoretical values of q/Q that led Prof. Lea and Dr. Tadros to suggest the presence of a turbulent core at the wire.

(9) Experiments were also made for the 0·047 in. wire in a pipe of diameter 0·976 in. and also for a rubber core of diameter 0·167 in. in the same pipe. This rubber core was threaded on a steel cable of such a diameter that the density of the composite wire was closely equal to that of water, so that sagging of the core when submerged in the water was eliminated.

The values of q/Q obtained from these experiments and also that obtained from the experiments with the $\frac{5}{8}$ in. diameter pipe are plotted against D/d in fig. 1. These experimental values fall closely on the curve predicted by the theory of the stream-line flow. The experimental values

Fig. 6.



of q/Q obtained by Prof. Lea and Dr. Tadros are plotted in the same figure.

(10) The gradual departure from linearity of the part BC of the curve ABC of fig. 5 indicates that some departure of stream-line flow is caused by the wire. To throw light on the nature of this departure the flow in the $\frac{5}{8}$ in. pipe with the 0·047 in. wire was examined with the ultra-microscope. The Reynolds number selected was

$$Vm/\nu = 380 \quad (\log_{10} Vm/\nu = 2{\cdot}58).$$

This value is outside the region in which the sudden breakdown occurs, although the departure from stream-line flow has become appreciable (see fig. 5).

To prevent astigmatic distortion of the view due to the pipe acting as a cylindrical lens the section under observation was submerged in a water-cell built round the outside of the glass tube at a distance of about 85 D from the mouth. The water in the pipe was illuminated through the reef surface of the water in the cell. Except for this modification the method of illumination and observation was the same as that described elsewhere *.

The impression formed from an examination of the flow near the wire was that slight turbulence was present and that the departure from rectilinear motion was too small to be measured with any accuracy. It was estimated that the maximum angular deviation from rectilinear flow was of the order of 1°. This slight departure from rectilinear motion was also observable at a distance from the wire, but the accuracy of the observations was not good enough to establish the relative degrees of turbulence near and at some distance from the wire. It is fairly certain, however, that any such difference is much too small to justify a sharp discrimination between the character of the flow near the wire and that in the body of the fluid.

In connexion with the gradual departure from stream-line flow observed with the wire (fig. 5, BC) it should perhaps be mentioned that a very similar effect was found in some earlier experiments on the pipe *without* a wire, which was traced to imperfect entry conditions—in particular to too small a supply tank. It may be that imperfections in the alignment or straightness of the wire are responsible for the slight departure from stream-line flow observed in the present experiments.

LVII. *The Change in the Infra-red Reflecting Power of Bismuth in a Magnetic Field.* *By* Prof. J. C. McLennan, *F.R.S.*, E. J. Allin, *Ph.D.*, *and* A. C. Burton, *Ph.D.* (*National Research Council Fellow, Canada*) †.

THE electromagnetic theory of light leads to a very definite relationship between the conductivity of a metal and its infra-red reflecting power (1). This relationship was tested experimentally by Hagen and Rubens (2), who used wave-lengths greater than 4μ and the direct current

* Proc. Roy. Soc. A, cxxxv. (1932).

† Communicated by the Authors.

conductivity. The agreement between the experimental results and those predicted by the theory was found by them to be fairly good for all metals studied with the single exception of bismuth. This exception, it was suggested, might be explained by assuming that the conductivity of bismuth for an alternating current is considerably different from its conductivity for a direct current, whereas for the other metals the conductivity is practically independent of the frequency of the applied field. These same investigators also studied the change in the reflecting power of different metals when a change in conductivity was brought about by an increase in temperature. Here, again, only in the case of bismuth, for which, instead of the increase in reflecting power looked for, a decrease comparable in magnitude was observed, was there a marked discrepancy between the values expected and those actually obtained.

Bismuth, it is well known, behaves very differently from other metals in another respect. It is possible by the application of even moderately high magnetic fields to change the resistance of bismuth to direct currents by as much as 50 per cent., the corresponding change in other metals being very small. As we had the apparatus available, we thought it would be of interest to investigate whether or not the infra-red reflecting power of bismuth changed when its direct-current resistance was changed by means of a magnetic field.

A narrow rectangular mirror of solid bismuth about 1 cm. in width was mounted between the rectangular pole pieces of a large electromagnet, and its resistance to direct current measured by a potentiometer both for zero field and for a field of several thousand gauss. The change in resistance produced by the field under the conditions chosen for the experiment was 36 per cent. The mirror and magnet were placed in such a position that the light from an incandescent glowbar was reflected upon a concave mirror, which in turn focussed it on the slit of a Hilger infra-red spectrometer. Care was taken to ensure that only the light reflected from the bismuth mirror reached the spectrometer.

Since it was important that there should be no shift of the mirror due to the field a preliminary experiment was carried out in which the image was allowed to fall upon a scale, its position being noted both before and after the field was applied. The wooden support on which the mirror was mounted was so wedged between the magnet pole pieces that these observations showed no movement of the image.

The galvanometer deflexion corresponding to zero field was

first obtained, and the spot of light watched for any motion when the field circuit was suddenly made. Readings were taken for wave-lengths 5 μ, 6 μ, 7 μ, and 8 μ. Since the observations were being made in the wave-length region beyond that of maximum intensity, the intensity of the radiation from the glowbar diminished as higher wave-lengths were reached, and the sensitivity of the method accordingly decreased. A movement of $\frac{1}{2}$ mm. of the galvanometer spot could have been easily detected, the corresponding change in the reflecting power at different wave-lengths being as follows :

5 μ—·3 per cent., 6 μ—1·4 per cent., 7 μ—1·6 per cent., 8 μ—1·9 per cent.

For no wave-length was there any detectable change in the galvanometer deflexion when the field was applied. Thus it was concluded that if there was any change in the reflecting power for these wave-lengths when the direct-current resistance was changed by 36 per cent. by a magnetic field the change was less than the values listed above.

The reflecting power of the mirror was measured as follows :—The glowbar was supported on an arm which could be rotated, the mirror being mounted above the centre of rotation. A diaphragm just smaller than the mirror was mounted directly in front of it, and the light from the mirror fell as before on a concave mirror which focussed it on the slit of the spectrometer. The galvanometer deflexion corresponding to this disposition of the apparatus was noted. The mirror was then removed, but the diaphragm left in place, and the arm rotated until the beam of light falling on the concave mirror occupied the same position as before. The galvanometer deflexion was again noted. The ratio of these two readings gave as the reflecting power of the mirror :

5 μ—49 per cent.	7 μ—45 per cent.
6 μ—46 per cent.	8 μ—46 per cent.

Since the relation connecting reflecting power and conductivity is of the form

$$(100-\mathrm{R})\sqrt{\mathrm{K}}=\text{constant},$$

where R is the reflecting power and K the conductivity, we should, assuming the change in conductivity produced by the field to be that given by the direct current readings, expect in our experiment, according to theory, a change in 100—R of 18 per cent. Using the value of R found above this would

correspond to a change in R of 21 per cent. No change comparable with this was, as we have seen, observed.

This value of the reflecting power is, however, very low. This may be partly due to experimental error, but is probably largely due to the nature of the mirror employed. Hagen and Rubens also state that they found mirrors of cast bismuth suitable only for experiments on changes in reflecting power. For determinations of absolute reflecting power they were forced to use mirrors formed by cathodic sputtering. It was at first intended to use such mirrors in these experiments, but the difficulty of making good electrical contacts with them led to their abandonment and the adoption of mirrors of cast bismuth. If instead of the value of R found above we use the value found by Hagen and Rubens for the cathodically sputtered mirrors we have :

$$4\mu\ (100-\mathrm{R}) = 24{\cdot}8,$$
$$8\mu\ (100-\mathrm{R}) = 18{\cdot}5.$$

A change of 18 per cent. in $100-\mathrm{R}$ would now correspond to a change in R of 4·9 per cent. at 4μ and of 4·1 per cent. at 8μ. These values also are very much greater than the smallest which could have been detected by us.

Thus it is seen that the effect of a magnetic field on the infra-red reflecting power of bismuth is very much less than the electromagnetic theory and the direct current value of the change in conductivity would lead us to expect. A possible explanation, suggested in part by the earlier work, is that the effect of the magnetic field on the conductivity for alternating currents is much less than its effect on the conductivity for direct currents. Attention might also be directed to the divergence between these results and those of which mention was made above, obtained when the change in conductivity was produced by a change in temperature.

All of the above observations suggest that an investigation of the resistance of bismuth for high-frequency currents and of the change in this resistance when the bismuth is placed in a magnetic field or when it is heated would lead to interesting results. A few such measurements in a magnetic field were carried out by Lenard [4] in 1890, but his experiments were not extensive enough to give us the desired information. The authors hope, with present-day facilities for generating and observing oscillating fields of different frequencies, to be able to carry out in the near future a complete set of experiments covering both the change of resistance with frequency and the change of resistance

caused by a magnetic field or an increase in temperature, the frequency being kept constant.

NOTE.—I desire to take this opportunity of expressing my gratitude to the National Research Council of Canada for the support given me, over a long term of years, through financial grants in aid of research and through awards to my collaborating students of bursaries, studentships, and fellowships.—J. C. M.

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McLennan Laboratory,
Department of Physics,
University of Toronto.
June 22nd, 1932.

LVIII. *A Formal Solution of Differential Equations by Series of Functions.* *By* W. H. McCREA, *M.A., Ph.D., Lecturer in Mathematics, University of Edinburgh* *.

Solution by Definite Integrals.

1. PROFESSOR E. T. WHITTAKER[1] has recently given a general theorem on the solution of linear differential equations by definite integrals. The object of this paper is to point out that an analogous theorem holds for the solution of differential equations by series of functions.

Whittaker's theorem may be briefly stated in the form: If a contact transformation from the variables q, p to the variables Q, P is given by $Q=\mathrm{Q}(q,p)$, $P=\mathrm{P}(q,p)$, and if we interpret p, P respectively as the operators $-\partial/\partial q$, $\partial/\partial \mathrm{Q}$, then if the function $\phi(Q)$ satisfies a given linear differential equation

$$\mathrm{G}(Q,P)\phi=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and if the function $\chi(q, Q)$ satisfies the pair of partial differential equations

$$[\mathrm{Q}(q,p)-Q]\chi=0, \quad [\mathrm{P}(q,p)-P]\chi=0, \quad . \quad (2)$$

* Communicated by the Author.

then the function

$$\psi(q)=\int\chi(q, Q)\phi(Q)dQ, \quad . \quad . \quad . \quad . \quad (3)$$

with a suitable path of integration, is a solution of the differential equation

$$\mathrm{G}\{\mathrm{Q}(q,p), \quad -\mathrm{P}(q,p)\}\psi=0 ; \quad . \quad . \quad . \quad (4)$$

that is, say,

$$\mathrm{F}(q,p)\psi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In two subsequent papers[2] W. O. Kermack and the author have given rules for obtaining equations (2) in compatible form, and for passing from the operator $\mathrm{G}(Q, P)$ to the operator $\mathrm{F}(q, p)$, and conversely. They have also developed the non-commutative algebra applicable to these operators.

Solution in Series of Functions.

2. We consider now the case where the argument Q of the function $\phi(Q)$ takes a set of discrete values with unit interval. We may write formally

$$\phi(Q+1)=e^{P}\phi(Q),$$

so that any linear function of $\phi(Q)$ and its differences may be written as a function of Q and e^{P} operating on $\phi(Q)$. For example, the function $\phi(Q)=z^{Q}/\Gamma(Q)$ satisfies the recurrence relation

$$Q\phi(Q+1)-z\phi(Q)=0.$$

This we shall write as

$$[Qe^{P}-z]\phi=0.$$

The theorem now stated is:—*If* $\phi(Q)$ *satisfies a given linear difference equation*

$$\mathrm{K}(Q, e^{P})\phi=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and if $\chi(q, Q)$ *satisfies a pair of compatible differential-difference equations*

$$[\mathrm{Q}(q,p)-Q]\chi=0, \quad [\mathrm{S}(q,p)-e^{P}]\chi=0, \quad . \quad (7)$$

then the series

$$\psi(q)=\Sigma_{Q}\chi(q, Q)\phi(Q), \quad . \quad . \quad . \quad . \quad (8)$$

summed over a suitable range of Q, *is a solution of the differential equation*

$$\mathrm{K}\{\mathrm{Q}(q,p), \quad \mathrm{S}^{-1}(q,p)\}\psi=0, \quad . \quad . \quad . \quad (9)$$

that is, say,

$$\mathrm{H}(q,p)\psi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Any pair of compatible equations for $\chi(q, Q)$, in which the variable Q and the operator e^P are not necessarily separated, as in (7), may be used in place of (7), provided the rules laid down in I § 2 for solving for Q, e^P be observed. The rules for relating $\mathrm{K}(Q, e^P)$ and $\mathrm{H}(q, p)$ are exactly those of I §§ 3.1, 3.2, to which reference should be made, since the procedure is not altogether simple. We may, however, summarize them by saying that *if* $\chi(q, Q)$ *satisfies* (7) *we pass from* K *to* H *by substituting* $\mathrm{Q}(q, p)$ *for* Q and $\mathrm{S}^{-1}(q, p)$ *for* e^P, while *if* $\chi(q, Q)$ *satisfies a pair of equations of the form*

$$[q-\mathrm{q}(Q, e^P)]\chi=0, \quad [p-\mathrm{p}(Q, e^P)]\chi=0 \quad . \quad (11)$$

then we pass from H *to* K *by substituting for* q, p *respectively the operators adjoint to* $\mathrm{q}(Q, e^P)$, $\mathrm{p}(Q, e^P)$.

Whittaker's theorem does not itself give the limits of the definite integral (3), but these must be determined by additional considerations. So in our case the range of summation must be fixed by additional considerations once the *form* of the result has been found from the general theorem.

We proceed to give some examples of the theorem. In the case of a formal result such as ours this is, perhaps, of more use than attempting a general proof. Such a proof would be analogous to those of Paper I. It is not suggested that the following methods are of practical value for obtaining these particular results ; but they appear to be instructive as illustrating the general relation brought to light by the theorem between a series with a given "kernel" function $\chi(q, Q)$ and its sum.

In § 7 a method of proof is sketched which depends on an application of Whittaker's theorem together with some further work of Kermack and the writer.

Examples involving Bessel Functions.

3. Consider $\chi(q, Q)=e^{iQq}$, where Q is integral. This satisfies, corresponding to (7), the equations

$$(ip-Q)\chi=0, \quad (e^{iq}-e^P)\chi=0.$$

Now $\mathrm{J}_n(z)$, the Bessel coefficient of order n, is known to satisfy the difference equation

$$\mathrm{J}_{n-1}(z)+\mathrm{J}_{n+1}(z)=\frac{2n}{z}\mathrm{J}_n(z).$$

If therefore we take $\phi(Q) = J_Q(z)$, it satisfies corresponding to (6) the equation

$$\left[e^{-P} + e^{P} - \frac{2Q}{z}\right]\phi = 0. \quad . \quad . \quad . \quad . \quad (12)$$

Hence by the theorem the function

$$\psi(q) = \Sigma_Q \chi(q, Q)\phi(Q)$$

satisfies an equation corresponding to (9), obtained by substituting ip for Q and e^{-iq} for e^P in (12). We find

$$\left[e^{iq} + e^{-iq} - \frac{2ip}{z}\right]\psi = 0,$$

or

$$\frac{d\psi}{dq} - iz\cos q \,.\, \psi = 0,$$

giving

$$\psi(q) = A \,.\, e^{iz\sin q}. \quad (A = \text{constant.})$$

Hence the relation

$$\psi(q) = \Sigma_Q \chi(q, Q)\phi(Q)$$

shows that there must hold a result of the form

$$Ae^{iz\sin q} = \Sigma_Q e^{iQq} J_Q(z).$$

This is the familiar property of Bessel coefficients, namely

$$e^{iz\sin\zeta} = \sum_{-\infty}^{\infty} J_n(z) e^{in\zeta}.$$

4. The example just given starts with a known function ϕ and proceeds to find the function ψ; so it is virtually a method of summing the given series. As an example of the converse process of obtaining an expansion for a given function we shall derive the known addition formula for the Bessel functions.

We shall look for an expansion of the form

$$J_m(q + \alpha) = \Sigma_Q J_Q(q)\phi(Q), \quad . \quad . \quad . \quad . \quad (13)$$

so that we take $\chi(q, Q) = J_Q(q)$, where Q is integral. As a pair of compatible equations for $\chi(q, Q)$ we can take any pair of the recurrence relations for $J_Q(q)$, the most convenient being

$$J_{Q-1}(q) + J_{Q+1}(q) = \frac{2Q}{q} J_Q(q),$$

$$J_Q'(q) = \tfrac{1}{2}\{J_{Q-1}(q) - J_{Q+1}(q)\}.$$

In our notation these become, corresponding now to equations (11),

$$[q-2(e^{-P}+e^{P})^{-1}Q]\chi=0,$$
$$[p-\tfrac{1}{2}(e^{P}-e^{-P})]\chi=0.$$

Now we have

$$\psi(q)=\mathrm{J}_m(q+\alpha),$$

which satisfies Bessel's equation

$$\frac{d^2\psi}{dq^2}+\frac{1}{q+\alpha}\frac{d\psi}{dq}+\left(1-\frac{m^2}{(q+\alpha)^2}\right)\psi=0,$$

or

$$[(q+\alpha)p(q+\alpha)p+(q+\alpha)^2-m^2]\psi=0. \quad . \quad (14)$$

By our theorem, and the rules quoted, we infer that $\phi(Q)$ satisfies a difference equation obtained by substituting in (14) for q the operator adjoint to

$$2(e^{-P}+e^{P})^{-1}Q, \text{ that is, } 2Q(e^{P}+e^{-P})^{-1},$$

and for p the operator adjoint to

$$\tfrac{1}{2}(e^{P}-e^{-P}), \text{ that is, } \tfrac{1}{2}(e^{-P}-e^{P}).$$

This gives

$$\{\tfrac{1}{4}[2Q(e^{P}+e^{-P})^{-1}+\alpha][e^{-P}-e^{P}][2Q(e^{P}+e^{-P})^{-1}+\alpha]$$
$$[e^{-P}-e^{P}]+[2Q(e^{P}+e^{-P})^{-1}+\alpha][2Q(e^{P}+e^{-P})^{-1}+\alpha]$$
$$-m^2\}\phi=0,$$

which reduces without difficulty to

$$\left[Q+m+\frac{\alpha}{2}(e^{P}+e^{-P})\right]\left[Q-m+\frac{\alpha}{2}(e^{P}+e^{-P})\right]\phi=0.$$

Hence $\phi(Q)$ satisfies one of the difference relations

$$\phi(Q+1)+\phi(Q-1)=\frac{2(\pm m-Q)}{\alpha}\cdot\phi(Q),$$

which are seen to be those satisfied by Bessel functions ; so that we have

$$\phi(Q)=\mathrm{J}_{\pm m-Q}(\alpha),$$

apart from a constant factor. It is evident that the positive sign of m must be taken. The theorem has thus shown that the expansion (13) is of the form

$$\mathrm{J}_m(q+\alpha)=\Sigma_Q\mathrm{J}_Q(q)\mathrm{J}_{m-Q}(\alpha).$$

The known result is of course

$$\mathrm{J}_m(q+\alpha)=\sum_{\nu=-\infty}^{\infty}\mathrm{J}_\nu(q)\mathrm{J}_{m-\nu}(\alpha).$$

Example involving Legendre Functions.

5. The theorem yields similarly the form of Heine's expansion of $(q-z)^{-1}$ in Legendre functions[3]. Here we must take $\chi(q, Q)=Q_Q(q)$, Legendre's function * of the second kind of integral order $Q=n$, say, and must look for an expansion

$$(q-z)^{-1}=\Sigma_n Q_n(q)\phi(n). \quad . \quad . \quad . \quad . \quad (15)$$

A suitable pair of compatible equations for χ are provided by the recurrence relations

$$(n+1)Q_{n+1}(q)-(2n+1)qQ_n(q)+nQ_{n-1}(q)=0,$$
$$Q'_{n+1}(q)-Q'_{n-1}(q)=(2n+1)Q_n(q),$$

which in our notation are

$$\{q-(2Q+1)^{-1}[(Q+1)e^P+Qe^{-P}]\}\chi=0,$$
$$\{p+(e^P-e^{-P})^{-1}(2Q+1)\}\chi=0.$$

Now the function $\psi(q)=(q-z)^{-1}$ satisfies the differential equation

$$(qp-zp-1)\psi=0. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Applying our rules as before, the equation for $\phi(Q)$ is got from (16) by substituting for q the operator adjoint to

$$(2Q+1)^{-1}[(Q+1)e^P+Qe^{-P}],$$

and for p the operator adjoint to

$$-(e^P-e^{-P})^{-1}(2Q+1).$$

This gives

$$\{[e^{-P}(Q+1)+e^PQ](2Q+1)^{-1}(2Q+1)(e^P-e^{-P})^{-1}$$
$$-z(2Q+1)(e^P-e^{-P})^{-1}-1\}\phi=0,$$

or $$[Qe^P-z(2Q+1)+(Q+1)e^{-P}](e^P-e^{-P})^{-1}\phi=0. \quad (17)$$

Now the difference equation

$$[Qe^P-z(2Q+1)+(Q+1)e^{-P}]\theta=0$$

is satisfied by $\theta(Q)=dP_Q(z)/dz$, where $P_Q(z)$ is the Legendre coefficient of order Q. So in (17) we may take

$$(e^P-e^{-P})^{-1}\phi=P_Q'(z),$$

or
$$\phi(Q)=P'_{Q+1}(z)-P'_{Q-1}(z)$$
$$=(2Q+1)P_Q(z).$$

* The use in this paragraph of Q, P, with suffixes, as the usual notation for Legendre functions should not lead to confusion with the symbols Q, P of the rest of the paper.

Consequently the relation (15) given by the theorem is of the form

$$(q-z)^{-1}=\Sigma_n(2n+1)\mathrm{P}_n(z)\mathrm{Q}_n(q).$$

Actually the precise form of Heine's result is

$$(q-z)^{-1}=\sum_{n=0}^{\infty}(2n+1)\mathrm{P}_n(z)\mathrm{Q}_n(q).$$

It is to be observed that since the functions $\mathrm{P}_n(z)$, $\mathrm{Q}_n(z)$ satisfy the same differential and difference equations, our theorem contains in itself no indication as to which should be taken. If it is found possible to develop an operational calculus which takes full account of boundary conditions, the right solution should be given automatically. As it is, however, we have already indicated that, after the theorem has given the form of the result sought, we have still to apply further considerations to fix certain details.

Example involving Parabolic Cylinder Functions.

6. Finally, as an example involving parabolic cylinder functions $\mathrm{D}_n(q)$, we shall evaluate

$$\psi(q)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\mathrm{D}_n(q)$$
$$=\Sigma_Q\phi(Q)\chi(q,Q),\text{ say.}\quad\ldots\quad(18)$$

The function $\mathrm{D}_n(q)$ satisfies the differential and difference equations [4]

$$\frac{d^2\mathrm{D}_n}{dq^2}+(n+\tfrac{1}{2}-\tfrac{1}{4}q^2)\,\mathrm{D}_n(q)=0,$$
$$\mathrm{D}_{n+1}(q)-\tfrac{1}{2}q\,\mathrm{D}_n(q)+\mathrm{D}_n'(q)=0,$$

which we write as

$$[p^2+\tfrac{1}{2}-\tfrac{1}{4}q^2+Q]\chi=0,$$
$$[p+\tfrac{1}{2}q-e^P]\chi=0.$$

The function $\phi(Q)=z^Q/Q!$ satisfies the difference equation

$$(ze^{-P}-Q)\phi=0.\quad\ldots\quad(19)$$

According to our rules the equation for $\psi(q)$ is therefore obtained from (19) by substituting

$$-(p^2+\tfrac{1}{2}-\tfrac{1}{4}q^2)\text{ for }Q,\text{ and }(p+\tfrac{1}{2}q)\text{ for }e^{-P},$$

giving

$$\left(p^2+zp+\frac{1}{2}+\frac{z}{2}q-\frac{1}{4}q^2\right)\psi=0,$$

or

$$\frac{d^2\psi}{dq^2}-z\frac{d\psi}{dq}+\left(\frac{1}{2}+\frac{z}{2}q-\frac{q^2}{4}\right)\psi=0.$$

It is readily found that the required solution is

$$\psi(q)=\mathrm{A}\exp(zq-\tfrac{1}{4}q^2),$$

where A is a constant. The precise result (18), when A has been adjusted, is

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} D_n(q) = \exp\left(-\tfrac{1}{2}z^2 + zq - \tfrac{1}{4}q^2\right).$$

This is a particular case of a general addition formula, susceptible of easy direct proof, namely,

$$D_m(q+\alpha) = \left[D_m(q) - \alpha D_{m+1}(q) + \frac{\alpha^2}{2!} D_{m+2}(q) - \ldots\right] \exp\left(\frac{\alpha^2}{4} + \frac{\alpha}{2} q\right).$$

Outline of Proof depending on Whittaker's Theorem.

7. It is interesting to establish a further connexion with Whittaker's theorem from another point of view.

Suppose that to the functions $\chi(q, Q)$ of § 2 there correspond a set of orthogonal functions $\chi^*(q, Q)$, such that

$$\int_C \chi(q, Q)\chi^*(q, Q')dq = \delta_{QQ'} \quad (\delta_{QQ}=1,\ \delta_{QQ'}=0,\ Q\neq Q'). \quad (20$$

for a known path of integration C. Then, if we multiply both sides of (8) by $\chi^*(q, Q')$, and integrate over C, we get formally

$$\phi(Q) = \int_C \chi^*(q, Q)\psi(q)dq. \quad . \quad . \quad . \quad (21)$$

Now it can be shown [5] that, *if* $\chi(q, Q)$ *satisfies the equations* (7), *then* $\chi^*(q, Q)$ *satisfies the adjoint equations*

$$[Q^*(q,p) - Q]\chi^* = 0, \quad [S^*(q,p) - e^{-P}]\chi^* = 0, \quad (22)$$

where $Q^*(q,p)$, $S^*(q,p)$ *are respectively the operators adjoint to* $Q(q,p)$, $S(q,p)$; *and if* $\chi(q, Q)$ *satisfies equations* (11), *then* χ^* *satisfies the adjoint equations*

$$[q - q^*(Q, e^P)]\chi^* = 0, \quad [p + p^*(Q, e^P)]\chi^* = 0, \quad (23)$$

where $q^*(Q, e^P)$, $p^*(Q, e^P)$ *are respectively the operators adjoint to* $q(Q, e^P)$, $p(Q, e^P)$.

But now we have here just the conditions of Whittaker's theorem as stated in § 1. We assume that C is a suitable path of integration, and we apply the theorem when required to difference equations regarded as differential equations of infinite order. Equation (21) corresponds to equation (3). We have the rôles of ϕ, ψ and of q, Q interchanged. Consequently equations (23) correspond to (2). Also the equation satisfied by $\psi(q)$, namely,

$$H(q, p)\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

corresponds to (1), while the equation satisfied by $\phi(Q)$, namely,

$$K(Q, e^P)\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

corresponds to (5). The laws connecting the operators H, K are still those given in Paper I., but now the rôles of H, K are different from § 2.

Whittaker's theorem, together with these laws, shows that *if* $H(q, p)$ *is given, then* K *is obtained*, taking account of (23), *by substituting for* q, p *the operators* $q^*(Q, e^P)$, $p^*(Q, e^P)$ *respectively*. Alternatively, *if* $K(Q, e^P)$ *is given, then* H *is obtained*, taking account of (22), by substituting for Q, e^{-P} the adjoint operators of $Q^*(q, p)$, $S^*(q, p)$, i. e., *by substituting for* Q, e^P *the operators* $Q(q, p)$, $S^{-1}(q, p)$. Hence in both cases *H, K are related in precisely the manner of* § 2.

Thus, when the orthogonal functions $\chi^*(q, Q)$ exist, our theorem amounts to an application of Whittaker's to determine the coefficients $\phi(Q)$ in an expansion of $\psi(q)$ in a series of the functions $\chi(q, Q)$ of the form $\Sigma_Q \phi(Q)\chi(q, Q)$.

References.

(1) E. T. Whittaker, Proc. Edinb. Math. Soc. (2) ii. pp. 189–204 (1931).
(2) W. O. Kermack and W. H. McCrea, *ibid.* pp. 205–219, 220–239. Referred to as Papers I., II. Reference may also be made to Kermack and McCrea, Proc. Roy. Soc. Edinb. li. pp. 176–189 (1931), where a completely different method of obtaining series of functions is given.
(3) Whittaker and Watson, 'Modern Analysis,' p. 321 (1927).
(4) *Ibid.* pp. 347–350.
(5) W. O. Kermack and W. H. McCrea. To be published elsewhere.

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LIX. *On the Buckling of a Rod naturally Curved.* By Prof. A. S. Lokshin, *Mining Institute at Dnepropetrovsk**.

§ 1.

THE buckling of a rod naturally curved is a very interesting problem for technics, and some authors have considered it† ; but it has not been sufficiently investigated.

* Communicated by the Author.

† During the last few years the following papers have appeared in which this problem has been discussed:—E. Chwalla, *Sitzungsberichte Wien*, cxxxvi. pp. 645 678 (1927); M. Mesnager, *Genie Civil*, 1929, p. 114; Batigle, *Genie Civil*, 1929, p. 113; Pigeau, *Genie Civil*, 1929, n. 19 and 20; E. Shtayerman, 'Visti' of pol. inst. at Kief, 1929, i. p. 25.

In this paper we start from the general equations of equilibrium*

$$\frac{dN}{ds}+T\kappa+X=0, \quad \frac{dT}{ds}-N\kappa+Z=0, \quad \frac{dM}{ds}+N=0, \quad (1)$$

where N = the shearing force, T the tension, M the bending moment, and κ the curvature of the axis. The increase of the curvature and the bending moment are bounded by the formula

$$M=B(\kappa-\kappa_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) become, on substitution from (2) and introduction of ρ_0 and θ,

$$\frac{dN}{d\theta}+T\left(1+\rho_0\frac{M}{B}\right)+X\rho_0=0,$$

$$\frac{dT}{d\theta}-N\left(1+\rho_0\frac{M}{B}\right)+\rho_0 Z=0, \quad \frac{dM}{d\theta}+\rho_0 N=0. \quad . \quad (3)$$

Differentiating with respect to θ the first of these equations, and inserting the value of $\frac{dT}{d\theta}$, we have

$$\left(\frac{d^2}{d\theta^2}+1\right)N+\rho_0 N\frac{M}{B}+\frac{1}{B}\cdot\frac{d}{d\theta}\rho_0 TM+\frac{d}{d\theta}(\rho_0 X)-\rho_0 Z=0.$$

The equation for the bending moment becomes, on omission of terms of higher order,

$$B\left(\frac{d^2}{d\theta^2}+1\right)\frac{1}{\rho_0}\frac{dM}{d\theta}+\frac{d}{d\theta}(\rho_0^2 XM)=B\left[\frac{d}{d\theta}(\rho_0 X)-\rho_0 Z\right]. \quad (4)$$

The displacement u is directed along the normal inwards, and the displacement w is directed along the tangent in the sense in which θ increases. Then the increase of the curvature

$$\kappa-\kappa_0=\frac{1}{\rho_0}\cdot\frac{d}{d\theta}\frac{w+\frac{du}{d\theta}}{\rho_0}.$$

We shall introduce the condition that the central line is unextended

$$\frac{dw}{d\theta}-u=0, \quad \text{whence } M=\frac{B}{\rho_0}\cdot\frac{d}{d\theta}\frac{1}{\rho_0}\left(\frac{a^2w}{d\theta^2}+w\right). \quad . \quad (5)$$

* A. E. H. Love, 'A Treatise on the Mathematical Theory of Elasticity,' 1927, p. 397.

In the particular case where $\rho_0 = a$ (a is constant) we obtain

$$M = \frac{B}{a^2}\left(\frac{d^3w}{d\theta^3} + \frac{dw}{d\theta}\right) \quad \text{or} \quad M = \frac{B}{a^2}\left(\frac{d^2u}{d\theta^2} + u\right). \quad . \quad (6)$$

This formula (6) is applicable by considering the bending of the circular rod. After one integration (5) follows

$$\frac{d^2w}{d\theta^2} + w = \frac{\rho_0}{B}\int_0^\theta \rho_0 M d\theta + c\rho_0 ; \quad . \quad . \quad . \quad (7)$$

c denotes a constant of integration. The general integral of (7) is

$$w = c_1 \cos\theta + c_2 \sin\theta$$
$$+ \frac{1}{B}\left\{ \sin\theta \int_0^\theta \rho_0 \cos\theta \, d\theta \int_0^\theta \rho_0 M \, d\theta \right.$$
$$\left. - \cos\theta \int_0^\theta \rho_0 \sin\theta \, d\theta \int_0^\theta \rho_0 M \, d\theta \right\}$$
$$+ c\left(\sin\theta \int_0^\theta \rho_0 \cos\theta \, d\theta - \cos\theta \int_0^\theta \rho_0 \sin\theta \, d\theta\right).$$

Here three constants appear, and by the integration (4) we shall again have three constants. Therefore, to determine the constants we must have six boundary conditions. If the ends of the rod are hinged the terminal conditions are

$$\text{at} \qquad \theta = 0, \quad w = 0, \quad \frac{dw}{d\theta} = 0, \quad M = 0,$$

$$\text{at} \qquad \theta = \gamma, \quad w = 0, \quad \frac{dw}{d\theta} = 0, \quad M = 0.$$

From four of these conditions we obtain

$$c_1 = c_2 = 0 \quad \text{and} \quad \frac{\int_0^\gamma \rho_0 \sin\theta \, d\theta \int_0^\theta \rho_0 M \, d\theta}{\int_0^\gamma \rho_0 \cos\theta \, d\theta \int_0^\theta \rho_0 M \, d\theta} = \frac{\int_0^\gamma \rho_0 \sin\theta \, d\theta}{\int_0^\gamma \rho_0 \cos\theta \, d\theta}. \quad (8)$$

For the case of the rod having ends hinged we have to integrate the equation (4) by the conditions at $\begin{matrix}\theta = 0, & M = 0 \\ \theta = \gamma, & M = 0\end{matrix}$ and the condition (8). They give three equations to determine

the constants of integration (4). When the ends of the rod are clamped the boundary conditions are

$$\text{at} \qquad \theta=0, \quad w=0, \quad \frac{dw}{d\theta}=0, \quad \frac{d^2w}{d\theta^2}=0,$$

$$\text{at} \qquad \theta=\gamma, \quad w=0, \quad \frac{dw}{d\theta}=0, \quad \frac{d^2w}{d\theta^2}=0,$$

whence

$$c=0, \; c_1=0, \; c_2=0,$$

$$\int_0^\gamma \rho_0 \cos\theta \, d\theta \int_0^\theta \rho_0 \mathrm{M} \, d\theta = 0, \quad \int_0^\gamma \rho_0 \sin\theta \, d\theta \int_0^\theta \rho_0 \mathrm{M} d\theta = 0,$$

$$\int_0^\gamma \rho_0 \mathrm{M} \, d\theta = 0.$$

The general integral of equation (4) is

$$\mathrm{M}=\mathrm{M}_0+\mathrm{A}_1\mathrm{M}_1+\mathrm{A}_2\mathrm{M}_2+\mathrm{A}_3\mathrm{M}_3,$$

where M_0 is a particular integral of the complete equation and M_1, M_2, and M_3 are independent particular integrals of the homogeneous one. If the ends are hinged the equations for determining the constants will be

$$\mathrm{A}_1\mathrm{M}_1(0)+\mathrm{A}_2\mathrm{M}_2(0)+\mathrm{A}_3\mathrm{M}_3(0)=-\mathrm{M}_0(0),$$
$$\mathrm{A}_1\mathrm{M}_1(\gamma)+\mathrm{A}_2\mathrm{M}_2(\gamma)+\mathrm{A}_3\mathrm{M}_3(\gamma)=-\mathrm{M}_0(\gamma),$$
$$\mathrm{A}_1\mathrm{V}_1+\mathrm{A}_2\mathrm{V}_2+\mathrm{A}_3\mathrm{V}_3=-\mathrm{V}_0,$$

where

$$\mathrm{V}_i=\int_0^\gamma \rho_0 \cos\theta \, d\theta \cdot \int_0^\gamma \rho_0 \sin\theta \, d\theta \int_0^\theta \rho_0 \mathrm{M}_i \, d\theta$$
$$-\int_0^\gamma \rho_0 \sin\theta \, d\theta \cdot \int_0^\gamma \rho_0 \cos\theta \, d\theta \int_0^\theta \rho_0 \mathrm{M}_i \, d\theta.$$

When the determinant

$$\Delta=\begin{vmatrix} \mathrm{M}_1(0), & \mathrm{M}_2(0), & \mathrm{M}_3(0) \\ \mathrm{M}_1(\gamma), & \mathrm{M}_2(\gamma), & \mathrm{M}_3(\gamma) \\ \mathrm{V}_1, & \mathrm{V}_2, & \mathrm{V}_3 \end{vmatrix} \quad . \quad . \quad . \quad (9)$$

is not equal to zero, we shall obtain determinate values of the constants. By the critical value of the loading, $\Delta=0$. Let us take the equation

$$\mathrm{B}\left(\frac{d^2}{d\theta^2}+1\right)\frac{1}{\rho_0}\frac{d\mathrm{G}}{d\theta}+\frac{d}{d\theta}\rho_0^2\mathrm{X}\mathrm{G}=0 \quad . \quad . \quad (10)$$

and find the value of the load for an integral of (10) which satisfies the conditions $G(0)=0$, $G(\gamma)=0$,

$$\int_0^\gamma \rho_0 \cos\theta\, d\theta \cdot \int_0^\gamma \rho_0 \sin\theta\, d\theta \int_0^\theta \rho_0 G\, d\theta - \int_0^\gamma \rho_0 \sin\theta\, d\theta \cdot \int_0^\gamma \rho_0 \cos\theta\, d\theta \int_0^\theta \rho_0 G\, d\theta = 0$$

and differs from zero. Remembering that

$$G = A_1 M_1 + A_2 M_2 + A_3 M_3,$$

we have for the load above mentioned

$$\begin{vmatrix} M_1(0), & M_2(0), & M_3(0) \\ M_1(\gamma), & M_2(\gamma), & M_3(\gamma) \\ V_1, & V_2, & V_3 \end{vmatrix} = 0.$$

In both cases the same equation is obtained. Therefore to find the critical load we proceed in the following manner. We reject in the complete equation its right hand part and integrate the equation from the conditions which are given for the bending moment. It is possible to show that this method may be applied to the other types of fastening.

If $$\frac{d}{d\theta}(\rho_0 X) - \rho_0 Z \neq 0,$$

the equation (10) is approximate.

§ 2. *Circular Rod.*

Let the central line be a circle of radius a; then M satisfies the equation

$$\frac{B}{a^3}\left(\frac{d^2}{d\theta^2}+1\right)\frac{dM}{d\theta} + \frac{d}{d\theta} XM = Ba\left[\frac{d}{d\theta} X - Z\right]. \quad . \quad (11)$$

The equations for determining the arbitrary constants will be obtained from the conditions

$$M(0)=0, \qquad M(\gamma)=0,$$

and $$\int_0^\gamma [\cos(\theta-\alpha) - \cos\alpha]\, M\, d\theta = 0, \quad . \quad . \quad (12)$$

where α denotes the half of the angle between the extreme normals. By the critical value the equation

$$\frac{B}{a^3}\left(\frac{d^2}{d\theta^2}+1\right)\frac{dG}{d\theta} + \frac{d}{d\theta} XG = 0. \quad . \quad . \quad . \quad (13)$$

has an integral not equal to zero, which satisfies the conditions

$$G(0)=0, \qquad G(\gamma)=0,$$

and $$\int_0^{\gamma}[\cos(\theta-\alpha)-\cos\alpha]\,G\,d\theta=0.$$

We shall introduce $\zeta=\theta-\alpha$ in the equation (13). If the load of uniform intensity p is normal to the axis we obtain

$$\frac{B}{a^3}\left(\frac{d^2}{d\zeta^2}+1\right)\frac{dG}{d\zeta}+p\frac{dG}{d\zeta}=0. \quad . \quad . \quad . \quad (14)$$

Taking $G=A\sin\frac{\pi\zeta}{\alpha}$, we satisfy the three conditions (12). From the equation (14) follows

$$p_\kappa=\frac{B}{a^3}\left(\frac{\pi^2}{\alpha^2}-1\right).$$

In the case where the uniformly distributed load is parallel to the axis of symmetry the equation (13) becomes

$$\frac{B}{a^3}\left(\frac{d^2}{d\zeta^2}+1\right)\frac{dG}{d\zeta}+p\frac{d}{d\zeta}\cos\zeta G=0, \quad . \quad . \quad (15)$$

where p denotes the intensity of the load. All conditions satisfy an odd function found from the equation (15) and equal to zero at $\zeta=\alpha$. For this function the boundary conditions are

at $\zeta=0, \quad G=0, \quad \frac{d^2G}{d\zeta^2}=0,$

at $\zeta=\alpha, \quad G=0.$

The first integration gives

$$\frac{B}{a^3}\left(\frac{d^2}{d\zeta^2}+1\right)G+p\cos\zeta.G=c.$$

From the boundary conditions it follows that $c=0$. Therefore we have to integrate the equation

$$\frac{B}{a^3}\left(\frac{d^2}{d\zeta^2}+1\right)G+p\cos\zeta G=0$$

by the boundary conditions $G(0)=0$ and $G(\alpha)=0$. Let m denote $\frac{pa^3}{B}$, then

$$\frac{d^2G}{d\zeta^2}+(1+m\cos\zeta)G=0. \quad . \quad . \quad . \quad . \quad (16)$$

To integrate Mathieu's equation (16) the approximate method is applied. We put

$$G = \Sigma A_n \sin \frac{n\pi\zeta}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and insert in the equation (16).

Expanding $\cos\zeta \sin \frac{n\pi\zeta}{\alpha}$ in a series of sines, and equating the coefficients of the sines to zero, we have a system of linear equations :

$$\left.\begin{aligned}
&A_1\left[1-\frac{\pi^2}{\alpha^2}+m\frac{\sin\alpha}{\alpha}\left(1+\frac{1}{\frac{4\pi^2}{\alpha^2}-1}\right)\right] \\
&\quad +m\frac{\sin\alpha}{\alpha}\Sigma(-1)^{i}_{i=2,3,4,5\ldots} A_i\left[\frac{1}{\frac{(i-1)^2\pi^2}{\alpha^2}-1}\right. \\
&\qquad\qquad \left.-\frac{1}{\frac{(i+1)^2\pi^2}{\alpha^2}-1}\right]=0, \\
&A_2\left[1-\frac{4\pi^2}{\alpha^2}+m\frac{\sin\alpha}{\alpha}\left(1+\frac{1}{\frac{16\pi^2}{\alpha^2}-1}\right)\right] \\
&\quad +m\frac{\sin\alpha}{\alpha}\Sigma(-1)^{i+1}_{i=1,3,4,5\ldots} A_i\left[\frac{1}{\frac{(i-2)^2\pi^2}{\alpha^2}-1}\right. \\
&\qquad\qquad \left.-\frac{1}{\frac{(1+2)^2\pi^2}{\alpha^2}-1}\right]=0, \\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&A_n\left[1-\frac{n^2\pi^2}{\alpha^2}+m\frac{\sin\alpha}{\alpha}\left(1+\frac{1}{\frac{n^2\pi^2}{\alpha^2}-1}\right)\right] \\
&\quad +m\frac{\sin\alpha}{\alpha}\Sigma(-1)^{i+n-1}_{i=1,2,3\ldots,n-1,n+1} A_i\left[\frac{1}{\frac{(i-n)^2\pi^2}{\alpha^2}-1}\right. \\
&\qquad\qquad \left.-\frac{1}{\frac{(i+n)^2\pi^2}{\alpha^2}-1}\right]=0.
\end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

Taking only two terms of the series (18) we get an approximate value of m:

$$m \approx 2 \frac{\alpha}{\sin \alpha} \frac{\frac{\pi^2}{\alpha^2} - 1}{\beta},$$

where

$$\beta = 1 + \frac{1}{\frac{4\pi^2}{\alpha^2} - 1} - \left(1 + \frac{1}{\frac{16\pi^2}{\alpha^2} - 1}\right) \frac{\pi^2 - \alpha^2}{4\pi^2 - \alpha^2}$$

$$+ \sqrt{\left[1 + \frac{1}{\frac{4\pi^2}{\alpha^2} - 1} - \left(1 + \frac{1}{\frac{16\pi^2}{\alpha^2} - 1}\right) \frac{\pi^2 - \alpha^2}{4\pi^2 - \alpha^2}\right]^2}$$

$$- 4 \left[\left(1 + \frac{1}{\frac{4\pi^2}{\alpha^2} - 1}\right)\left(1 + \frac{1}{\frac{16\pi^2}{\alpha^2} - 1}\right)\right.$$

$$\left. - \left(\frac{1}{\frac{\pi^2}{\alpha^2} - 1} - \frac{1}{\frac{9\pi^2}{\alpha^2} - 1}\right)\right]^2 \frac{\pi^2 - \alpha^2}{4\pi^2 - \alpha^2};$$

. . . (19)

for instance, if $\alpha = 0{\cdot}5$, $m = 39{\cdot}89$.

§ 3. *A Circular Rod on Elastic Foundation.*

At first we shall consider the buckling of a complete circular ring when the load is uniformly distributed and normal to the axis. For this case $X = p - \kappa_1 u$, where p denotes the intensity of the load and κ_1 is the coefficient of the foundation. The displacement u satisfies the equation

$$\frac{B}{a^3}\left(\frac{d^2}{d\theta^2} + 1\right)\frac{d}{d\theta}\left(\frac{d^2u}{d\theta^2} + u\right) + p\frac{d}{d\theta}\left(\frac{d^2u}{d\theta^2} + u\right) + \kappa_1 a \frac{du}{d\theta} = 0. \quad (20)$$

We take $u = A \sin 2\theta$ and insert in the equation (20). The critical value $p_\kappa = \frac{3B}{a^3} + \frac{\kappa a}{3}$. For the case of an incomplete rod the condition $u = \frac{dw}{d\theta}$ gives

$$\frac{B}{a^3}\left(\frac{d^2}{d\theta^2} + 1\right)\frac{d^2}{d\theta^2}\left(\frac{d^2w}{d\theta^2} + w\right) + p\frac{d^2}{d\theta^2}\left(\frac{d^2w}{d\theta^2} + w\right) + \kappa_1 a \frac{d^2w}{d\theta^2} = 0. \quad (21)$$

If the ends are hinged the boundary conditions are :

$$\text{at} \quad \left.\begin{array}{llll} \theta=0, & w=0, & \dfrac{dw}{d\theta}=0, & \dfrac{d^3w}{d\theta^3}=0, \\ \theta=\gamma, & w=0, & \dfrac{dw}{d\theta}=0, & \dfrac{d^3w}{d\theta^3}=0. \end{array}\right\} \quad . \quad . \quad (22)$$

Taking

$$w=\mathrm{A}\left(1-\cos\frac{2n\pi}{\gamma}\theta\right)^{*},$$

we satisfy the conditions (22). The critical value is given by the formula

$$p_\kappa=\frac{\mathrm{B}}{a^3}\left(\frac{4n^2\pi^2}{\gamma^2}-1\right)+\frac{\kappa_1 a}{\dfrac{4n^2\pi^2}{\gamma^2}-1}. \quad . \quad . \quad (23)$$

When $$\kappa_1=\frac{\mathrm{B}\left(\dfrac{4\pi^2}{\gamma^2}-1\right)\left(\dfrac{16\pi^2}{\gamma^2}-1\right)}{a^4}$$

the same values are obtained for $n=1$ and $n=2$. If κ_1 is somewhat smaller than this quantity we find p_κ by taking $n=1$, when it is somewhat greater $n=2$.

The terminal value by which we should take $n+1$ instead of n is

$$\kappa_1=\frac{\mathrm{B}\left[\dfrac{4\pi^2}{\gamma^2}(n+1)^2-1\right]\left(\dfrac{4n^2\pi^2}{\gamma^2}-1\right)}{a^4}.$$

§ 7. *The Parabolic Arch.*

We consider a parabolic arch having hinged ends. For a parabola $\dfrac{1}{\rho_0}=\dfrac{1}{a}\cos^3\zeta$, where a denotes the radius of curvature at $\zeta=0$. The equation for the bending moment becomes

$$\frac{\mathrm{B}}{a^3}\left(\frac{d^2}{d\zeta^2}+1\right)\cos^3\zeta\frac{d\mathrm{M}}{d\zeta}+\frac{d}{d\zeta}\frac{\mathrm{X}}{\cos^6\zeta}\mathrm{M}=\frac{\mathrm{B}}{a}\left[\frac{d}{d\zeta}\left(\frac{\mathrm{X}}{\cos^3\zeta}\right)-\frac{\mathrm{Z}}{\cos^3\zeta}\right] \quad . \quad . \quad . \quad (24)$$

From the conditions at the ends we have

$$\mathrm{M}(-\alpha)=0, \qquad \mathrm{M}(\alpha)=0,$$

and $$\int_{-\alpha}^{\alpha}\left(\frac{1}{\cos^2\alpha}-\frac{1}{\cos^2\zeta}\right)\frac{\mathrm{M}}{\cos^3\zeta}d\zeta=0.$$

* n is integer.

In order to obtain the critical value of the load it is necessary to integrate the equation

$$\frac{B}{a^3}\left(\frac{d^2}{d\zeta^2}+1\right)\cos^3\zeta\frac{dG}{d\zeta}+\frac{d}{d\zeta}\frac{X}{\cos^6\zeta}G=0 \quad . \quad . \quad (25)$$

with the conditions

$$G(-\alpha)=0, \qquad G(\alpha)=0,$$

and

$$\int_{-\alpha}^{+\alpha}\left(\frac{1}{\cos^2\alpha}-\frac{1}{\cos^2\zeta}\right)\frac{G}{\cos^3\zeta}d\zeta=0. \quad . \quad . \quad (26)$$

To find an odd function which satisfies equation (25) and the conditions (26) we take

$$\text{at} \qquad \zeta=0, \quad G=0, \quad \frac{d^2G}{d\zeta^2}=0,$$

$$\zeta=\alpha, \quad G=0.$$

When the load is uniformly distributed and parallel to the axis of symmetry $X=p\cos\zeta$, and equation (25) becomes

$$\frac{B}{a^3}\left(\frac{d^2}{d\zeta^2}+1\right)\cos^3\zeta\frac{dG}{d\zeta}+p\frac{d}{d\zeta}\frac{G}{\cos^5\zeta}=0. \quad . \quad (27)$$

We put $\cos^3\zeta\dfrac{dG}{d\zeta}=W$, and integrate the system of equations

$$\left(\frac{d^2}{d\zeta^2}+1\right)W+\frac{m}{\cos^6\zeta}\left(\frac{W}{\cos^2\zeta}+5\sin\zeta.G\right)=0, \quad \frac{dG}{d\zeta}=\frac{W}{\cos^3\zeta},$$

with the boundary conditions at

$$\zeta=0, \quad W=1, \quad G=0, \quad \frac{d^2G}{d\zeta^2}=0, \quad m=\frac{pa^3}{B}.$$

Applying the Störmer-Adams * method we obtain

$$\Delta G_n=\eta_n+\frac{1}{2}\Delta\eta_{n-1}+\frac{5}{12}\Delta^2\eta_{n-2}+\frac{3}{8}\Delta^3\eta_{n-3},$$

$$\Delta W_{n-1}=\xi_n+\frac{1}{12}\left[\Delta^2\xi_{n-2}+\Delta^3\xi_{n-3}+\Delta^4\xi_{n-4}-\frac{1}{20}\Delta^4\xi_{n-4}\right],$$

where

$$\eta=h\frac{W}{\cos^3\zeta} \text{ and } \xi=-h^2\left[W+\frac{m}{\cos^6\zeta}\left(\frac{W}{\cos^2\zeta}+5\sin\zeta.G\right)\right].$$

* A. Krylof, 'The Approximate Integration of Differential Equations,' 1923.

At the critical p, $G(\alpha)=0$. By using this method we find the approximate value of m. For instance, for $\alpha=0,5$, $m\approx 26{\cdot}87$.

Dnepropetrovsk.
19th January, 1932.

LX. *Notices respecting New Books.*

The Methods of Statistics. By L. H. C. Tippett, M.Sc. [8vo, pp. 221.] (Williams & Norgate. Price 15*s.*)

THE need for a knowledge of statistical method is none the less important to workers in the biological field, textiles, and agriculture than to those working in the relatively exact science of physics, and it is to all such workers that Mr. Tippett's most useful work is intended to appeal.

The treatment of the subject is full and adequate, and, being meant more for serious study than casual reference, every attempt has been made to define properly all the necessary technical terms.

The two early chapters are devoted to explaining the application of the probability laws to distribution curves, and the results (derived clearly and adequately) are applied later to the diverse needs of experimentalists, the whole being illustrated by a fascinating series of statistics, ranging from the relative heights of Englishmen and Scotsmen to the state of children's teeth in relation to their use of a tooth-brush. The theory of random sampling which covers the majority of the requirements of the straightforward worker is given in Chapter III. in sufficient detail to be of real use, and this is developed later in a chapter on small samples. From Chapter VII. onwards a very thorough treatment of correlation is given.

The arrangement and printing are excellent, and no experimenter should feel safe without a knowledge of the contents.

Vision and Colour Vision. By R. A. Houstoun. [Pp. 238, 102 diagrams.] (Longmans, Green & Co. Price 15*s.*)

Dr. Houstoun is well known as a writer of text-books on physical subjects, and his particular interest in colour and optics naturally explains the appearance of this latest effort.

A volume on this topic is particularly welcome at the present moment, in view of the fact that the new work of the last decade has not so far been collected in a book of this character.

The scope and aim of the book should bring it within the spheres of interest of physicists, physiologists, psychologists, as well as statisticians.

The writer, being a physicist, suitably opens his account of the subject with a chapter on Units, which has clearly been conceived in very proper sympathy for the general reader in that it is kept clear and simple, while graded type invites "time saving skipping" by the reader whose hope is to obtain general ideas rather than details.

This scheme is freely employed throughout, so that the book contains much more than would at first appear.

The next chapter, on intensity, leads up to the recent interesting work on dark adaptation, while the following two chapters deal intimately with acuity of vision and spectral visibility.

Chapters on colour mixing and hue discrimination follow, where to the well-known fundamental facts and laws are added more recently obtained data. It is in these chapters that the physicist and the psychologist perhaps approach closest. But the writer of the book is clearly aware of the pitfalls sometimes inherent in the possible simple interpretations of these really rather complex phenomena.

The physicist in particular would have liked to see a more definite explanation of the precise importance of the frequently quoted well-known formula $h\nu = \frac{1}{2}mv^2$ in reference to the photo-electric "explanation" of vision.

While in the opinion of the reviewer a glossary of terms would have enhanced the value of the book—it is a most useful collection of information brought together in an interesting and coherent way—the printing and numerous figures are all excellent.

LXI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

May 25th, 1932.—Sir Thomas H. Holland, K.C.S.I., K.C.I.E., D.Sc., F.R.S., President, in the Chair.

THE following communications were read:—

1. 'The Influence of Geological and Climatological Factors on the Distribution of non-marine Fossil Reptiles and Stegocephalia.' By Baron Ferencz von Nopcsa, Foreign Member. (Communicated by the Foreign Secretary.)

A study of the terrestrial tetrapods and their distribution indicates three lines of migration in Upper Palæozoic and Lower Mesozoic times: (1) North America to Europe, thence through India and the Malay Archipelago to Australia, and from there to South America; (2) Central Asia to South Africa; and (3) Africa to South America.

In Jurassic, Cretaceous, and Tertiary times there appear four lines: (1) Europe to Asia, the Malay Archipelago, and South America; (2) Asia, Fenno-Scandia, Greenland, and North America; (3) Central Asia to South Africa; and (4) North Africa to Brazil. To some extent these migratory lines are the same as those of Upper Palæozoic and Lower Mesozoic times.

An attempt is made to correlate them with large-scale tectonic movements of folding and marine transgression, together with continuous changes in the ecliptic, bringing about a cycle of climatic and seasonal changes.

2. 'The Geology of the Country between Abergwesyn (Breconshire) and Llansawel (Carmarthenshire).' By Kenneth Arthur Davies, M.Sc., Ph.D., F.G.S.

The country described comprises about 130 square miles, and lies on the north-western side of the Towy anticlinorium. The rocks belong wholly to the graptolitic facies, and are very similar to those in adjacent areas which have already been described. They range from the Bala, through the Birkhill to the Gala stage, and consist of shales and mudstones with subordinate grits and conglomerates. In contrast with most parts of Central Wales, arenaceous material is common at many horizons. An inlier of green and purple sediments also falls within the area described.

The base of the Middle Birkhill and the base of the Gala overlap the rocks below, though the latter transgression is to be seen over a greater area than the former.

Evidence of contemporaneous folding transverse to the subsequent Caledonian System is obtained here as in the areas where the shelly facies is developed.

It is suggested that the differences in grain are indicative of the varying depth of the sea-floor when deposition took place, the coarse deposits being laid down in banks tailing off from the areas made shallow by the cross-folding movements.

3. 'The Conglomerates and Grits of the Bala and Valentian Rocks of the District between Rhayader (Radnorshire) and Llansawel (Carmarthenshire).' By Kenneth Arthur Davies, M.Sc., Ph.D., F.G.S., and John Isaac Platt, M.Sc., F.G.S.

The grits and conglomerates discussed in this paper occur within the rocks between the Lower Gala and the Upper Bala stages in the region lying on the western side of the Towy anticlinorium, between Rhayader (Radnorshire) and Llansawel (Carmarthenshire). The sediments described occur within shales and mudstones of the graptolitic facies of the groups represented.

A description is given of the distribution and field characteristics of the arenaceous bands, as well as a petrographical description of the included pebbles of the conglomerates and of the heavy mineral constituents of the grits. Most of the pebbles are quartzite and vein-quartz; sandstones, shales, etc. also occur frequently, but numerous pebbles of igneous rocks were found, as well as a few metamorphic types. The majority of the igneous pebbles are acid volcanic rocks, but interesting intrusive types also occur. All the igneous rocks are rich in soda. Suggestions are made regarding the mode of deposition of the coarse deposits, and the possible source or sources of origin of the material is discussed.

June 8th, 1932.—Mr. W. Campbell Smith, M.C., T.D., M.A., Vice-President, in the Chair.

Mr. Lawrence Rickard Wager, B.A., B.Sc., F.G.S., delivered a lecture on 'Geological Work in East Greenland during the British Arctic Air-Route Expedition, 1930–31.'

The winter base of the Expedition was in the Angmagsalik district and most of the geological work was carried out there and also in the course of two journeys. The first journey was northwards for 300 miles along the coast to Kangerdlugsuak, and the second along the edge of the ice-cap to Mount Forel.

The Angmagsalik district consists of gneisses and schists for which the name Metamorphic Complex is used, since the Archæan age of these rocks cannot be proved. Of earlier age than the intermediate to acid gneiss with amphibolites which forms the bulk of the Metamorphic Complex are a suite of plutonic rocks resembling the Charnockite Series of India, and extensive rafts of highly metamorphosed and injected sediments. The strike of the Metamorphic Complex is dominantly north-west to south-east, and the folding does not, therefore, seem to be a continuation of the north-east Greenland belt of Caledonian mountains. Rocks later than the Metamorphic Complex are a few sheets of amphibolite and, in the neighbourhood of Angmagsalik Island, a series of massive gabbros, diorites, and granites, which differ from the Tertiary granites farther north and may be Caledonian in age.

Tertiary basalts and minor intrusions were known to occur along 250 miles of the coast in north-eastern Greenland and also in a big area south of Scoresby Sound. They have now been found to extend intermittently to Cape Holm, 120 miles south of their previously conjectured limit. Here the lower basalts are interbedded with a thin, thermally metamorphosed, sedimentary series in which were found poorly-preserved fossils suggesting a lower Tertiary age. Farther north in the Kangerdlugsuak area, a series of 300 feet of fine-grained tuffs,

which were laid down in water, rests on the Metamorphic Complex ; above these are pillow lavas of spilitic composition. The chief interest of the district, however, lies in the discovery of several centres of Tertiary plutonic intrusion at Kangerdlugsuak and for 120 miles southwards, where the lavas have been largely removed by erosion. A striking Tertiary dyke-swarm, parallel to the north-north-east and south-south-west trend of the coast, cuts the Metamorphic Complex and the basalts.

The coastal zone between the ice-cap and the sea north of Angmagsalik consists of high mountains and includes Mount Forel, the highest mountain within the Arctic. These mountains have been cut from a raised peneplain which passes without interruption from the Metamorphic Complex to the Tertiary basalts and plutonic intrusions. The peneplain has a gentle dip to the south-east or east-south-east, and to this is due the general trend of the coast and of one group of valleys and fjords. Another group is related to faulting and Tertiary dykes parallel to the north-north-east and south-south-west alinement of the coast.

The inland ice has recently receded, and its present action may be studied side by side with its effects on the recently uncovered mountainous surface. The Expedition had experience of a complete year of weather on the ice-cap, and if that year is typical there would seem to be now little or no accumulation of snow.

Mr. Wager exhibited rock-specimens and lantern-slides in illustration of his lecture.

June 22nd, 1932.—Sir Thomas H. Holland, K.C.S.I., K.C.I.E., D.Sc., F.R.S., President, in the Chair.

The following communications were read :—

1. 'A Revision of the British Carboniferous Members of the Family Loxonematidæ, with Descriptions of New Forms.' By Jane Longstaff (née Donald), F.L.S., F.G.S. (Read in abstract by the Secretary.)

Many Carboniferous species have been referred to the genus *Loxonema* (Phillips) which do not strictly belong to it. This is the case with Etheridge's Catalogue, where 25 species are given. Some of them, however, belong to the family Loxonematidæ as defined by Koken, and are referable to several of the genera into which it has since been divided. These are *Zygopleura* Koken, *Katosira* Koken, *Microptychis* Longstaff, and *Hemizyga* Girty. Three new genera are suggested for

other species. The genus *Zygopleura* Koken contains by far the largest number of the species ; one of these, *Z. rugifera* (Phillips), exhibits a considerable amount of variation both in size and ornamentation. The nuclear whorls, also, accord more with that genus than with *Pseudozygopleura* Knight.

Twenty-six species and several varieties are described, 13 of the former being new. Six had been previously noted by de Koninck as common to Belgium, and another is now recorded: namely, *L. subconstricta* de Koninck. That author has described a greater number of forms, but the British species are on the whole better preserved, many having the protoconch intact.

A table is given showing the general range in time and space. It is impossible to note definite horizons therein, but these are specified in as much detail as possible in the text.

2. 'The Dalradian Succession in Islay and Jura.' By Archibald Allison, M.A., B.Sc. (communicated by Prof. E. B. Bailey, M.C., B.A., F.R.S., F.G.S.).

A variety of opinion has been expressed regarding the structure and succession of the rocks of Islay and Jura. The Author outlines results recently obtained by himself through the application of the criteria of current-bedding and graded bedding, restricting his attention to the Dalradian rocks north and east of the Loch Skerrols thrust. Previously, this method of study has scarcely been used in Islay.

Two types of current-bedding have been observed, small scale and large scale. The former is a development of ripple marking and occurs in dolomites and flags of the Dolomitic Group, and also in sub-boulder bed dolomites belonging to the Portaskaig Conglomerate. Large-scale current-bedding of any value was seen only in the 'Pipe Rock', a siliceous member occurring near the base of the Dolomitic Group.

Along part of the east coast of Jura, and more locally on the same strike in Islay, graded bedding has been found, unaccompanied by current-bedding. It occurs in grits, some interbedded in the Jura Slates, others forming the main mass of the Scarba Conglomeratic Group.

In addition, the Author has carefully studied, and in one case sketched, erosion effects, to which previous authors have directed attention in the Portaskaig Conglomerate at Beannan Dubh in North Islay.

Altogether, the evidence of original order of succession is abundant, widely dispersed, and consistent. Combined with the fact that the Maol an Fhithich Quartzite appears naturally

to underlie the Mull of Oa Phyllites, it leads the Author to conclude that the structure of Islay is an anticline, in part steeply overturned towards the north-west, and that the original succession is, in descending order :—

(11) Ardmore and Laphroaig Quartzites.
(10) Port Ellen Phyllites.
(9) Scarba Conglomeratic Group.
(8) Jura Slates.
(7) Main Quartzite.
(6) Dolomitic Group of North Islay.
(5) Lower fine-grained quartzite of North Islay.
(4) Portaskaig Conglomerate.
(3) Islay Limestone.
(2) Mull of Oa Phyllites.
(1) Maol an Fhithich Quartzite.

This is in complete agreement with the views of Prof. E. B. Bailey as expounded in his paper on the Islay anticline (Q. J. G. S. vol. lxxii, 1917, pp. 132–59).

William Francis.

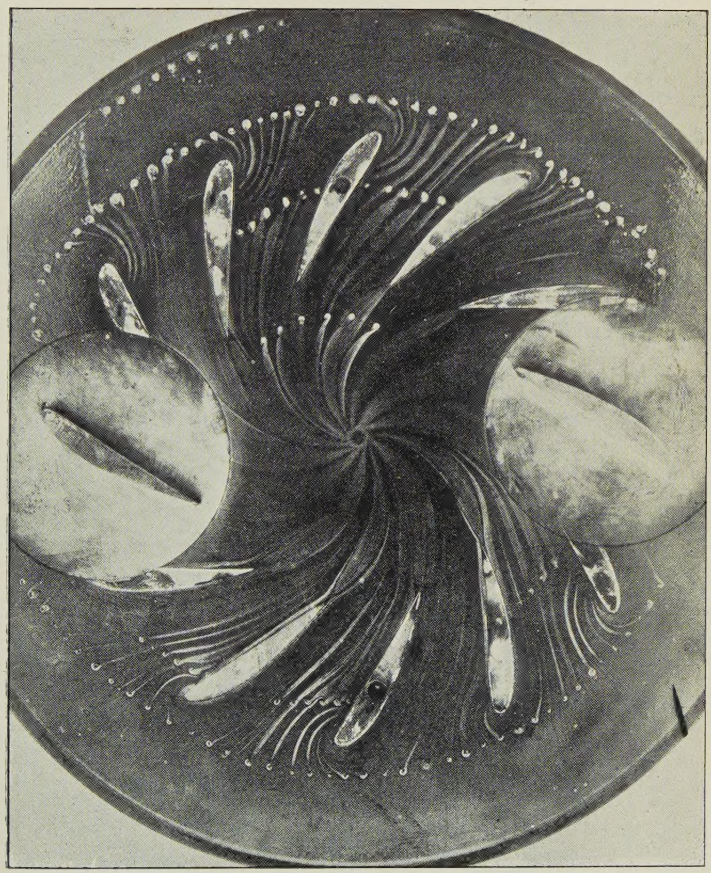
It is with very deep regret that we have to announce the death of Mr. William Francis, who passed away at Eastbourne on Tuesday the 16th August last, after a long and painful illness extending over many months. He was the elder son of Dr. William Francis, with whom he had been long associated in conducting this Magazine, taking over the greater part of the Editorial duties at his father's death in 1904. An account of his life and activities will appear in the October issue.

The Editors do not hold themselves responsible for the views expressed by their correspondents.

FIG. 2.

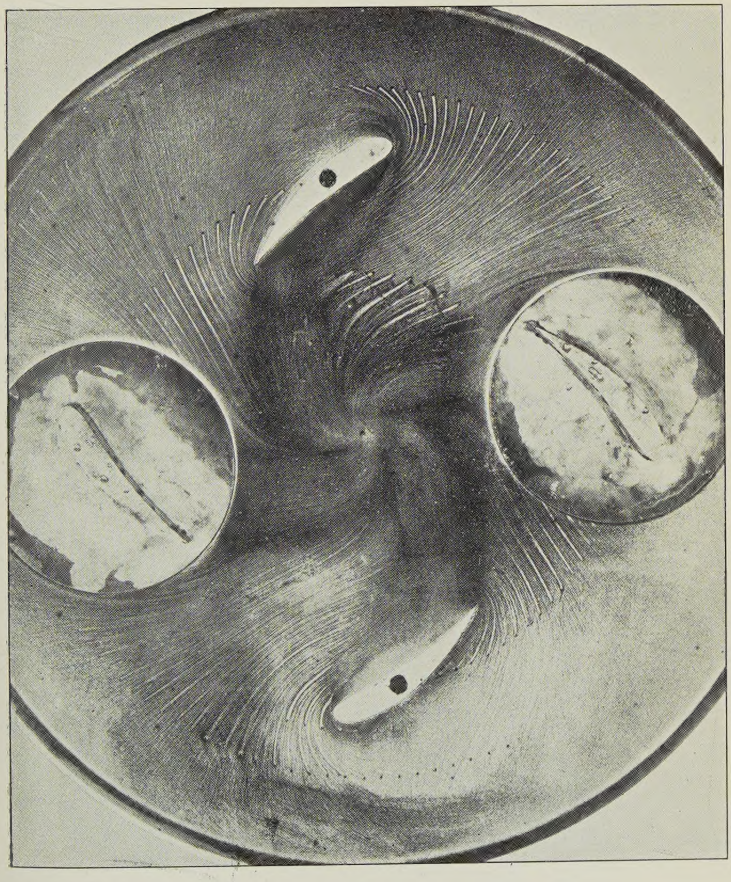
$t = 15.1^\circ$ C., $\alpha_1 = 40^\circ$, $v = 28$ cm./sec.
$\alpha_2 = 27.5^\circ$, $\Pi = 46.7$ mm.

FIG. 3.



$t = 10{\cdot}3°$ C. $a_2 = -56°$. $v_t = 30{\cdot}6$ cm./sec.
$a_p = -35°$. $H_m = 45{\cdot}7$ cm.

FIG. 4.



$t = 15{\cdot}1°$ C. $a_z = -86°$. $v_t = 43{\cdot}1$ cm./sec.
$a_p = -62°$. $H_m = 45{\cdot}8$ cm.